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MATHEMATICAL ANALYSIS IN QUESTIONS AND PROBLEMS

MATHEMATICAL ANALYSIS in Questions and Problems

Under the Editorship
of B.F. Butuzov

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Contents

Preface 7

List of Symbols 9

Chapter 1. Real Numbers 11

1.1. Comparison of Real Numbers 11
1.2. The Least Upper and the Greatest Lower Bound of a Set. Use of the Symbols of Mathematical Logic 14
1.3. Arithmetic Operations Involving Real Numbers 18
1.4. Induction 22

Chapter 2. Limit of a Sequence 24

2.1. Bounded and Unbounded Sequences. Limit of a Sequence 24
2.2. Infinitely Small and Infinitely Large Sequences 31
2.3. Properties of Convergent Sequences 35
2.4. Remarkable Examples 41
2.5. Monotonic Sequences 45
2.6. Limit Points 47
2.7. Fundamental Sequences. Cauchy Condition for Convergence of a Sequence 50

Chapter 3. Limit of a Function. Continuity of a Function 53

3.1. Limit of a Function. Limit Theorems. Infinitely Large Functions 53
3.2. Continuity of a Function at a Point 63
3.3. Comparison of Infinitesimal Functions. The Order Symbol "o" and Its Properties 69
3.4. Calculating the Limits of Functions by Means of Asymptotic Formulas. Calculating the Limits of Exponential Power Functions 76

Chapter 4. Derivatives and Differentials 85

4.1. The Derivative of a Function. Differentiation Rules 85
4.2. The Differential of a Function 100
4.3. Higher-Order Derivatives and Differentials 105

Chapter 5. The Indefinite Integral 112

5.1. The Antiderivative and the Indefinite Integral 112
5.2. Basic Indefinite Integrals 114
5.3. The Method of a Change of Variable 117

5.4. Integration by Parts 121
5.5. Integration of Rational Functions 125
5.6. Integration of Irrational Functions 129
5.7. Integration of Trigonometric Functions 134

Chapter 6. Fundamental Theorems on Continuous and Differentiable Functions 136

6.1. Theorems on the Boundedness of Continuous Functions 136
6.2. Uniform Continuity of a Function 142
6.3. Some Theorems on Differentiable Functions . 147
6.4. L'Hospital's Rule 154
6.5. Taylor's Formula 158

Chapter 7. Investigating the Behaviour of a Function and Constructing Graphs 163

7.1. Constructing the Graphs of Explicit Functions 163
7.2. Investigation of Plane Curves Represented Parametrically 173

Chapter 8. The Definite Integral 181

8.1. Integrability of a Function (in the Sense of Riemann) and the Definite Integral 181
8.2. Properties of the Definite Integral 189
8.3. Newton-Leibniz Formula 195
8.4. Calculating the Lengths of Plane Curves 209
8.5. Calculating the Areas of Plane Figures 213
8.6. Calculating the Volumes of Solids 218
8.7. Application of the Definite Integral in Physics 221

Chapter 9. Lebesgue Measure and Lebesgue Integral 226

9.1. The Measure of a Set 226
9.2. Measurable Functions 234
9.3. Lebesgue Integral 238

Answers and Hints 243
Index 267

Preface

This study aid is based on many years' experience of lecturing on mathematical analysis at the first course at the physics faculty of Moscow University. It is intended for students as well as for teachers, especially young ones, who are starting their lecturing career.

The book covers the analysis of functions of one variable, including the concepts of Lebesgue measure and Lebesgue integral. It is not a collection of problems in the ordinary sense. As can be seen from its structure, its aim is to help the student master the material both actively and informally. As a rule, the material in each section is divided into four subsections.

Subsection I, entitled "Fundamental Concepts and Theorems", presents the principal theory and formulas (without proof) necessary for problem solving. The definitions and theorems correspond, in most cases, to those given by V. A. Ilyin and E. G. Poznyak in *Fundamentals of Mathematical Analysis**, which is now used as the main analysis textbook at the physics faculty of Moscow University. Sometimes a definition or theorem is followed by examples or comments which should make it easier for the student to understand the new concepts. Wherever possible, we give a physical interpretation of mathematical concepts.

Subsection II, called "Control Questions and Assignments", contains questions on the theory of analysis and simple problems whose solution does not involve complicated computations but which illustrate the theory presented. The questions and assignments given in this subsection will help the student master the theory and check his knowledge of the material already covered. It should be remembered that it supplements the main textbook where all concepts and theorems are followed by explanations and proofs. However, to solve a problem,

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* V. A. Ilyin, E. G. Poznyak, *Fundamentals of Mathematical Analysis*, Mir Publishers, Moscow, 1982.

it is often sufficient to understand the gist of a theorem or formula. Many control problems are intended to show the student this gist. Teachers may use the questions given in subsection II to check the knowledge of their students on a particular theme.

Subsection III, entitled "Worked Problems", illustrates the practical application of the theory. Considerable attention is given not only to the techniques of problem solving but also to various tricks and subtleties, say, the conditions for applicability of a theorem or formula. The number of worked problems varies depending on the volume and importance of the material covered. Sometimes subsection III includes an answer to a question posed in subsection II.

The purpose of subsection IV, called "Problems and Exercises for Independent Work", is self-evident. These are only a few exercises but they are diverse. The authors used many sources for the problems and exercises, *Problems in Mathematical Analysis* under the editorship of B. P. Demidovich being one. Therefore many problems are not original although there are a number of new problems in the book. The problems and exercises in subsection IV are answered or hints are given at the end of the book.

The beginning and end of a solution are marked by the signs △ and ▲, respectively, and the sign ● is used instead of the word "hint".

We hope that the book will be useful to students in their studies of mathematical analysis. All critical remarks and wishes will be accepted with gratitude.

The authors

List of Symbols

$\mathbf{N}$	The set of all natural numbers
$\mathbf{Z}$	The set of all integers
$\mathbf{R}$	The set of all real numbers (the number line)
$[a, b]$	A closed interval
(a, b)	An open interval
$[a, b)$, $(a, b]$	A half-closed or half-open interval
$[a, +\infty)$, $(a, +\infty)$ $(-\infty, a]$, $(-\infty, -a)$	A half-line
$\exists x$	There is an x
$\forall x$	For any x
$x \in X$	The number x belongs to the set X
$x \notin X$	The number x does not belong to the set X
$A \cup B$	The union of the sets A and B
$A \cap B$	The intersection of the sets A and B
$A \setminus B$	The difference of the sets A and B
$\inf X$	The greatest lower bound of the set X
$\sup X$	The least upper bound of the set X
$\{x_n\}$, (x_n)	A number sequence
$\sum\limits_{k=1}^{\infty} a_k$	A number series
$\lim\limits_{n\to\infty} x_n = a$	The number a is a limit of the sequence $\{x_n\}$
$y_n \ll x_n$	An infinitely large sequence $\{x_n\}$ has a higher order of growth than the infinitely large sequence $\{y_n\}$
$\varliminf\limits_{n\to\infty} x_n$	The lower limit of the sequence $\{x_n\}$
$\varlimsup\limits_{n\to\infty} x_n$	The upper limit of the sequence $\{x_n\}$
$\lim\limits_{x\to a} f(x) = b$, $f(x) \to b$ as $x \to a$	The number b is a limit of the function $f(x)$ as x tends to a

$\lim\limits_{x\to a+0} f(x) = b$, $\lim\limits_{x\to a-0} f(x) = b$	The number b is a limit of the function $f(x)$ as x tends to a from the right (from the left)
$\lim\limits_{x\to a+0} f(x) = \infty$, $\lim\limits_{x\to a-0} f(x) = \infty$	The function $f(x)$ is infinitely large on the right (on the left) of the point a
$\inf\limits_X f(x)$	The greatest lower bound of the function $f(x)$ on the set X
$\sup\limits_X f(x)$	The least upper bound of the function $f(x)$ on the set X
$D(x)$	The Dirichlet function
$[x]$	The integral part of the number x
$\operatorname{sgn} x$	The signum function
$\alpha \sim \beta$ as $x \to a$	The infinitesimals α and β are equivalent as $x \to a$
$\alpha = o(\beta)$ as $x \to a$	$\alpha(x)$ is an infinitesimal of a higher order of smallness than (β) x as $x \to a$
Δy	The increment of a function at a point
$f'(x)$, $y'(x)$	The derivative of the function $y = f(x)$ at the point x
$f'(x+0)$, $f'(x-0)$	The right-hand (left-hand) derivative at the point x
$\mathrm{d}y$	The differential of the function $y(x)$
$f^{(n)}(x)$	The nth derivative of the function $f(x)$
$\mathrm{d}^n y$	The nth-order differential of the function $y(x)$
$\mathbf{r} = \mathbf{r}(t)$	A vector function
$\int f(x)\,\mathrm{d}x$	The indefinite integral of the function $f(x)$
$\int\limits_a^b f(x)\,\mathrm{d}x$	The definite integral of the function $f(x)$ over the interval $[a, b]$
$\int\limits_E f(x)\,\mathrm{d}\mu(x)$	The Lebesgue integral of the function $f(x)$ over the set E
$f(x) \cong g(x)$ on E	The functions $f(x)$ and $g(x)$ are equivalent on the measurable set E

Chapter 1

Real Numbers

1.1. Comparison of Real Numbers

I. Fundamental Concepts and Theorems

1. Representing real numbers as nonterminating decimal fractions. Any real number a can be represented as a nonterminating decimal fraction

$$a = \pm a_0.a_1a_2\ldots a_n\ldots,$$

where we choose one of the two signs "$\pm$": plus for positive numbers and minus for negative numbers (the plus sign is usually omitted).

Rational numbers can be represented as repeating and irrational numbers as nonrepeating nonterminating decimal fractions. Some rational numbers can be represented as a terminating fraction or, what is the same, as a nonterminating fraction with zero in the period. Numbers of this kind admit of another representation, as a nonterminating decimal fraction with the figure nine in the period. For instance,

$$1/2 = 0.500\ldots0\ldots = 0.5\overline{0}, \quad 1/2 = 0.4999\ldots9\ldots = 0.4\overline{9}.$$

When comparing real numbers, we shall use only the first form of notation for rational numbers of this kind (with zero in the period).

2. Rule of comparison of real numbers. Let $a = \pm a_0.a_1a_2\ldots a_n\ldots$ and $b = \pm b_0.b_1b_2\ldots b_n\ldots$ be arbitrary real numbers represented as nonterminating decimal fractions.

The numbers a and b are *equal* ($a = b$) if they are of the same sign and the equalities $a_k = b_k$ ($k = 0, 1, 2, \ldots$) hold true. Otherwise we assume that $a \neq b$.

When comparing unequal numbers a and b, we shall consider three cases:

(1) a and b are nonnegative numbers. Since $a \neq b$, there is a natural number n (or $n = 0$) such that $a_k = b_k$

$(k = 0, 1, \ldots, n-1)$ and $a_n \neq b_n$. We assume that $a > b$ if $a_n > b_n$ and $a < b$ if $a_n < b_n$;

(2) a is a nonnegative number and b is a negative number. We assume that $a > b$;

(3) a and b are negative numbers. We assume that $a > b$ if $|a| < |b|$ and $a < b$ if $|a| > |b|$.

3. Some number sets. Real numbers can be given as points on a coordinate line*. Therefore the set of all real numbers is called a *number line* (*number axis*) and the numbers are called *points*, and the geometric interpretation is often used when number sets are considered. We shall use the following designations and terminology:

N *is the set of all natural numbers*,

Z *is the set of all integers*,

$\mathbf{R} = (-\infty, +\infty)$ *is the set of all real numbers* (*the number line*),

$[a, b]$ *is a closed interval* or the set of all real numbers x which satisfy the inequalities $a \leqslant x \leqslant b$,

(a, b) *is an open interval* or the set of all real numbers x which satisfy the inequalities $a < x < b$,

$[a, b)$ or $(a, b]$ *is a half-open or half-closed interval* or the set of all real numbers x which satisfy the inequalities $a \leqslant x < b$, $a < x \leqslant b$ respectively,

$[a, +\infty)$ or $(a, +\infty)$ or $(-\infty, a]$ or $(-\infty, a)$ *is a half-line* or the set of all real numbers x which satisfy the inequalities $a \leqslant x < +\infty$, $a < x < +\infty$, $-\infty < x \leqslant a$, $-\infty < x < a$ respectively,

the *neighbourhood* of the point c is any interval which contains the point c,

the ε-*neighbourhood* of the point c is the interval $(c - \varepsilon, c + \varepsilon)$, where $\varepsilon > 0$.

II. Control Questions and Assignments

1. What is the difference between the nonterminating decimal fractions which represent rational and irrational numbers?

2. In what case are two numbers equal?

3. Do the equalities $0.41\overline{9} = 0.42\overline{0} = 0.42$ hold true?

4. Formulate the rule of comparison of two unequal numbers.

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* Recall that the coodinate line is a straight line on which an arbitrary reference point, called the origin, a unit of length and a positive direction are chosen.

III. Worked Problems

1. Prove that for any real numbers a and b $(a < b)$ there is a rational number c such that $a < c < b$.

△ Assume, for definiteness, that the numbers a and b are positive, i.e.

$$a = a_0.a_1a_2\ldots a_k\ldots > 0,$$
$$b = b_0.b_1b_2\ldots b_k\ldots > 0.$$

If any one of them is a rational number, which can be expressed as a fraction with a period 9, then we write it as a fraction with 0 in the period. By the hypothesis, $a < b$. This means that there is a nonnegative integer n such that $a_k = b_k$ $(k = 0, 1, \ldots, n-1)$ and $a_n < b_n$. Since the figure 9 is not a period of the number a, there is a natural number $i > n$ such that $a_i \neq 9$.

Consider a rational number $c = c_0.c_1c_2\ldots c_i$, where $c_k = a_k$ $(k = 0, 1, \ldots, i-1)$, $c_i = a_i + 1$. The number c is larger than a since $c_k = a_k$ $(k = 0, 1, \ldots, i-1)$, $c_i = a_i + 1 > a_i$, and smaller than b since $c_k = a_k = b_k$ $(k = 0, 1, \ldots, n-1)$, $c_n = a_n < b_n$. Thus there is a rational number c such that $a < c < b$. ▲

2. Prove that for any real numbers a and b $(a < b)$ there is an irrational number α such that $a < \alpha < b$.

△ On the assumption of example 1 we consider the number

$$\alpha = c_0.c_1c_2\ldots c_i01001\,0001\ldots\underbrace{000\ldots0}_{n\text{ zeros}}1\,\underbrace{000\ldots0}_{n+1\text{ zeros}}1\ldots$$

This is, evidently, a nonrepeating fraction (explain why), i.e. α is an irrational number. This number is larger than a since $c_k = a_k$ $(k = 0, 1, \ldots, i-1)$, $c_i = a_i + 1 > a_i$; and smaller than b since $c_k = b_k$ $(k = 0, 1, \ldots, n-1)$, $c_n = a_n < b_n$. Thus there is an irrational number α such that $a < \alpha < b$. ▲

IV. Problems and Exercises for Independent Work

1. Prove that $\sqrt{8}$ is an irrational number.
2. Represent the fraction $31.2\overline{88}$ as an ordinary fraction.
3. Prove that any repeating decimal fraction which does not have figure 9 in the period can be obtained as a result of division of two natural numbers.

4. Prove that any repeating decimal fraction which has figure 9 in the period can be obtained as a result of division of two natural numbers.

5. Prove that for any two real numbers a and b ($a \neq b$) there are infinitely many rational and irrational numbers between them.

6. Prove the transitivity of the signs "=", ">", i.e. (a) if $a = b$ and $b = c$, then $a = c$, (b) if $a > b$ and $b > c$, then $a > c$.

7. Prove that the inequalities $-|a| \leqslant a \leqslant |a|$ hold true for any number a.

8. Prove that if $x \leqslant y$, then $-x \geqslant -y$.

1.2. The Least Upper and the Greatest Lower Bound of a Set. The Use of the Symbols of Mathematical Logic

I. Fundamental Concepts and Theorems

1. On the use of some logical symbols. Let X be a non-empty set of real numbers.

Definition. *The set X is* ***bounded from above (from below)*** *if there is a number M (m) such that for any number x from the set X the inequality $x \leqslant M$ $(x \geqslant m)$ holds true. The number M (m) is the* ***upper*** *(the* ***lower***) ***bound*** *of the set X.*

In this definition and in the formulations of many other definitions and theorems use is made of the words "there is" and "for any". To make the notation shorter, we shall use the logical symbols $\exists$ and $\forall$ instead.

The symbol $\exists$ is the *existential quantifier* and the symbol $\forall$ is the *universal quantifier*. The fact that the number x belongs (does not belong) to the set X will be designated as $x \in X$ $(x \notin X)$.

Using these symbols, we can write the definition of a set bounded from above as follows: the set X is bounded from above if $\exists M \in \mathbf{R}$ such that $\forall x \in X$ the inequality $x \leqslant M$ holds true, or (even shorter, omitting certain words), the set X is bounded from above if

$$\exists M \in \mathbf{R} \quad \forall x \in X\colon\ x \leqslant M. \qquad (1)$$

The use of quantifiers not only makes the notation shorter but also allows us to construct negations of propositions

(definitions, assertions) in a simple way. We shall illustrate this technique by an example of the negation of the definition of a set bounded from above. In other words, we shall formulate the definition of a set unbounded from above. The unboundedness of the set X from above means that there is no number M such that for any $x \in X$ the inequality $x \leqslant M$ holds true. This means that for any number M there is $x \in X$ for which $x > M$. Using the quantifiers, we can write the definition of a set unbounded from above as follows: the set X is unbounded from above if

$$\forall M \in \mathbf{R}\ \exists x \in X: x > M. \tag{2}$$

Comparing (1) and (2), we see that to construct the negation of proposition (1), we must replace the quantifier $\exists$ by $\forall$ and the quantifier $\forall$ by $\exists$ and replace the inequality appearing after the colon sign by exactly the converse proposition.

This rule can also be used to construct negations of other statements containing the quantifiers $\exists$ and $\forall$.

2. The least upper bound and the greatest lower bound of number sets

Definition. *The number $\bar{x}$ is the **least upper bound** of the set X bounded from above if* (1^0) $\forall x \in X$: $x \leqslant \bar{x}$, (2^0) $\forall \tilde{x} < \bar{x}\ \exists x \in X$: $x > \tilde{x}$.

Condition (1^0) means that $\bar{x}$ is one of the upper bounds of the set X and condition (2^0) means that $\bar{x}$ is the least of the upper bounds of the set X, i.e. no number $\tilde{x}$, which is smaller than $\bar{x}$, is an upper bound. The least upper bound of the set X is designated as $\sup X$.

The greatest lower bound* of the set X bounded from below is defined by analogy and designated as $\inf X$.

Theorem. *A nonempty set bounded from above (from below) has the least upper (the greatest lower) bound.*

If the set X is not bounded from above (from below), then we write $\sup X = +\infty$ ($\inf X = -\infty$).

The set X is *bounded* if it is bounded both from above and from below, i.e.

$$\exists M, m \forall x \in X: m \leqslant x \leqslant M. \tag{3}$$

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* You may come across books on analysis where the least upper bound (the greatest lower bound) is simply called the upper (the lower) bound.

II. Control Questions and Assignments

1. Use quantifiers to write a definition of a set bounded from below. Construct the negation of this definition using the rule of constructing negations.
2. Give a definition of the least upper (the greatest lower) bound of a set bounded from above (from below).
3. Formulate the theorem on the existence of the least upper bound and the greatest lower bound of a number set.
4. Prove that the least upper bound and the greatest lower bound are unique, i.e. that a set bounded from above (from below) has only one least upper bound and one greatest lower bound.
5. Show that the least upper bound and the greatest lower bound may or may not belong to a set. Give examples of number sets X in which (a) $\sup X \in X$, (b) $\sup X \notin X$, (c) $\inf X \in X$, (d) $\inf X \notin X$. Does the number X have the greatest number in cases (a) and (b) and the least number in cases (c) and (d)?
6. What is the meaning of the symbolic notation (a) $\sup X = +\infty$, (b) $\inf X = -\infty$?
7. What set is said to be bounded?
8. Prove that the following definition of a bounded set is equivalent to that given in subsection I: the set X is bounded if $\exists A > 0\ \forall x \in X\colon |x| \leqslant A$.
9. Applying the rule of constructing negations to the definition given in assignment 8, formulate the definition of an unbounded set.

III. Worked Problems

1. Find the least upper bound of the interval $(0, 1)$.

$\triangle$ The number 1 is an upper bound of the interval $(0, 1)$ since $\forall x \in (0, 1)\colon x < 1$. Furthermore, $\forall \tilde{x} < 1\ \exists a \in (0, 1)\colon a > \tilde{x}$. Indeed, if $\tilde{x} \leqslant 0$, then $\forall a \in (0, 1)\colon a > x$. If $\tilde{x} > 0$, then, as was shown in Example 1 in 1.1, there is a rational number a on the interval $(\tilde{x}, 1)$ such that $\tilde{x} < a < 1$, i.e. $\exists a \in (0, 1)\colon a > \tilde{x}$. Thus, both conditions for the definition of the least upper bound are fulfilled for the number 1. Consequently, $\sup (0, 1) = 1$. Note that the least upper bound we have found does not belong to the interval $(0, 1)$, i.e. $\sup (0, 1) \notin (0, 1)$,

whereas for the interval $(0, 1]$ we have $\sup (0, 1] = 1 \in (0, 1]$. ▲

2. Find the least upper bound and the greatest lower bound of the set of all proper rational fractions m/n ($m, n \in \mathrm{N}$, $m < n$) and show that this set does not have either the least or the greatest element.

△ Let X be the set of all proper rational fractions m/n. Since $\forall m, n \in \mathrm{N}\ m/n > 0$, the number 0 is a lower bound of the set X. Furthermore,

$$\forall \tilde{x} > 0 \quad \exists a \in X: a < \tilde{x}. \tag{4}$$

Indeed, if $\tilde{x} \geqslant 1$, then the proper rational fraction $a = 1/2$ satisfies condition (4). If $0 < \tilde{x} < 1$, then the number $\tilde{x}$ can be written as a nonterminating decimal fraction

$$\tilde{x} = 0.x_1x_2\ldots x_k\ldots,$$

$\exists n$ being such that $x_n \neq 0$. According to the rule of comparison of real numbers, the rational number

$$a = 0.x_1x_2\ldots x_{n-1}\,(x_n - 1)1$$

satisfies the inequalities $0 < a < \tilde{x} < 1$, i.e. is a proper rational fraction and satisfies condition (4).

Thus the second condition of the definition of the greatest lower bound of a number set is fulfilled for the number 0, and so $\inf X = 0$.

Since the set X contains only proper fractions, i.e. $m < n$, we have $m/n < 1$. Hence the number 1 is an upper bound of the set X. Furthermore, $\forall \tilde{x} < 1\ \exists m/n \in X$: $m/n > \tilde{x}$. Indeed, as was shown in Example 1 in 1.1, there is a rational number x_1 such that $\tilde{x} < x_1 < 1$. Since $x_1 < 1$, it follows that x_1 is a proper fraction: $x_1 = m/n$ ($m < n$), i.e. $x_1 \in X$. Consequently, both conditions for the definition of the least upper bound of a number set are fulfilled for the number 1. Thus $\sup X = 1$.

However, $\inf X = 0 \notin X$ since $m/n = 0$ only for $m = 0$, but $0 \notin \mathrm{N}$. This means that the set X does not have the least element. Precisely for the same reason $\sup X = 1 \notin X$ since $m/n = 1$ only for $m = n$, and this

contradicts the requirement that the fraction should be proper. Hence the set X does not have the greatest element. ▲

IV. Problems and Exercises for Independent Work

9. Assume that X and Y be nonempty sets of real numbers, X is bounded from above and Y is contained in X. Prove that Y is also bounded from above and $\sup Y \leqslant \sup X$.

10. Find the least upper bound and the greatest lower bound of the set of rational numbers x which satisfy the inequality $x^2 < 2$.

11. Let A be a set of numbers which are opposite in sign to the numbers of the set B. Prove that (a) $\inf A = -\sup B$, (b) $\sup A = -\inf B$.

1.3. Arithmetic Operations Involving Real Numbers

I. Fundamental Concepts

1. Addition and multiplication of rational numbers. The following rules of addition and multiplication of rational numbers are known:

$$\frac{m_1}{n_1}+\frac{m_2}{n_2}=\frac{m_1n_2+m_2n_1}{n_1n_2}, \qquad \frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{m_1m_2}{n_1n_2}. \tag{1}$$

Let us define the operations of addition and multiplication involving any real numbers.

2. Addition of real numbers. Assume that x and y are arbitrary real numbers and x_1 and y_1 are any rational numbers which satisfy the inequalities

$$x_1 \leqslant x, \quad y_1 \leqslant y. \tag{2}$$

The symbol $(x_1 + y_1)_r$ will mean that the numbers x_1 and y_1 can be added according to rule (1) of addition of rational numbers. Let us consider the set $\{(x_1 + y_1)_r\}$ of various sums of rational numbers x_1 and y_1 which satisfy condition (2). This set is bounded from above and, consequently, possesses the least upper bound.

The *sum* of the real numbers x and y is $\sup \{(x_1 + y_1)_r\}$.

3. Multiplication of real numbers. Assume that x and y are arbitrary positive real numbers and x_1 and y_1 are any rational numbers which satisfy the inequalities $0 < x_1 \leqslant x$, $0 < y_1 \leqslant y$. The symbol $(x_1y_1)_r$ will mean that the numbers x_1 and y_1 can be multiplied according to rule (1) of multiplication of rational numbers. Let us consider the set $\{(x_1y_1)_r\}$ of various products of this kind of rational numbers. The set is bounded from above and, consequently, possesses the least upper bound.

The *product* of positive real numbers x and y is $\sup \{(x_1y_1)_r\}$.

The product of real numbers of any sign is defined by the following laws:

(1⁰) $x \cdot 0 = 0 \cdot x = 0$,

$$(2^0)\ xy = \begin{cases} |x| \cdot |y| & \text{if } x \text{ and } y \text{ are of like signs,} \\ -|x| \cdot |y| & \text{if } x \text{ and } y \text{ are of unlike signs.} \end{cases}$$

II. Control Questions and Assignments

1. Formulate the rules of addition and multiplication of any two real numbers. Prove that the sets $\{(x_1 + y_1)_r\}$ and $\{(x_1y_1)_r\}$, which appear in these rules, are bounded from above.

2. Prove that the rule of addition of real numbers possesses the following properties: (a) $x + y = y + x$ (commutativity), (b) $(x + y) + z = x + (y + z)$ (associativity).

3. Prove that the rule of multiplication of real numbers possesses the following properties: (a) $xy = yx$ (commutativity), (b) $(xy)\,z = x\,(yz)$ (associativity).

III. Worked Problems

1. Prove that addition of two rational numbers according to the rule of addition of real numbers yields the same result as the addition according to rule (1) for rational numbers.

$\triangle$ Assume that x and y are arbitrary rational numbers, $(x + y)_r$ is their sum obtained with the use of the rule of addition of rational numbers and $(x + y)$ is their sum obtained with the use of the rule of addition of real numbers, i.e. $x + y = \sup \{(x_1 + y_1)_r\}$, where x_1 and y_1 are any rational numbers which satisfy the inequalities

$x_1 \leqslant x$, $y_1 \leqslant y$. We must prove that $x + y = (x + y)_r$, or

$$\sup \{(x_1 + y_1)_r\} = (x + y)_r.$$

To do that, we must show, in accordance with the definition of the least upper bound of a set, that

(1⁰) $\forall (x_1+y_1)_r \in \{(x_1+y_1)_r\}$: $(x_1+y_1)_r \leqslant (x+y)_r$,

(2⁰) $\forall \tilde{x} < (x+y)_r \ \exists (x_1+y_1)_r \in \{(x_1+y_1)_r\}$: $(x_1+y_1)_r > \tilde{x}$.

Since $x_1 \leqslant x$ and $y_1 \leqslant y$, it follows that $(x_1 + y_1)_r \leqslant (x + y)_r$ (for rational numbers this property of inequalities is known). Consequently, condition 1⁰ is fulfilled. Let us show that condition 2⁰ is also fulfilled.

Let $\tilde{x}$ be an arbitrary number smaller than $(x + y)_r$. Between the numbers x and $(x + y)_r$ there is a rational number a (see Example 1 in 1.1) such that $x < a < (x + y)_r$. We set $\delta = (x + y)_r - a$ (the subtraction is carried out according to the rule of subtraction of rational numbers). Then $a = (x + y)_r - \delta$, and since $\delta > 0$, there is a natural number n such that $2/n < \delta$.

We shall consider now rational numbers $x_1 = x - \frac{1}{n}$ and $y_1 = y - \frac{1}{n}$. Since $x_1 < x$ and $y_1 < y$, it follows that $(x_1 + y_1)_r \in \{(x_1 + y_1)_r\}$, and then $(x_1 + y_1)_r = (x+y)_r - \frac{2}{n} > (x+y)_r - \delta = a$ since $\frac{2}{n} < \delta$.

Thus $(x_1 + y_1)_r > a > \tilde{x}$, i.e. $(x_1 + y_1)_r > \tilde{x}$. We have thus shown that condition 2⁰ is fulfilled. ▲

2. Prove that $\forall x$: $x + (-x) = 0$.

△ Let x_1 and y_1 be any rational numbers which satisfy the inequalities $x_1 \leqslant x$, $y_1 \leqslant -x$. We must prove that $\sup \{(x_1 + y_1)_r\} = 0$, i.e.

(1⁰) $\forall (x_1 + y_1)_r \in \{(x_1 + y_1)_r\}$: $(x_1 + y_1)_r \leqslant 0$,

(2⁰) $\forall \tilde{x} < 0 \ \exists (x_1 + y_1)_r \in \{(x_1 + y_1)_r\}$: $(x_1 + y_1)_r > \tilde{x}$.

Since $y_1 \leqslant -x$, it follows that $-y_1 \geqslant x$ (it is easy to establish this fact using the rule of comparison of real numbers, see Exercise 8 in 1.1). By virtue of transitivity of the sign "$\leqslant$", it follows from the inequalities $x_1 \leqslant x$ and $x \leqslant -y_1$ that $x_1 \leqslant -y_1$ and, hence, $(x_1 + y_1)_r \leqslant 0$. Thus condition 1⁰ is satisfied.

We shall show that condition 2^0 is also satisfied. Let $\tilde{x}$ be an arbitrary negative number. Since $-\tilde{x} > 0$, there is a natural n such that $1/10^n < -\tilde{x}$, i.e. $-1/10^n > \tilde{x}$. Let us represent the number x as a nonterminating decimal fraction (for definiteness, we assume that $x > 0$):

$$x = x_0.x_1x_2\ldots x_n\ldots .$$

It follows from the rule of comparison of real numbers that

$$x_1 = x_0.x_1x_2\ldots x_n \leqslant x,$$

$$y_1 = -x_0.x_1x_2\ldots x_n - \frac{1}{10^n} \leqslant -x.$$

Thus $(x_1 + y_1)_r \in \{(x_1 + y_1)_r\}$, and then $(x_1 + y_1)_r = -1/10^n > \tilde{x}$, i.e. we have proved that condition 2^0 is fulfilled. ▲

IV. Problems and Exercises for Independent Work

12. Prove that multiplication of two rational numbers according to the rule of multiplication of real numbers yields the same result as multiplication according to rule (1) of rational numbers.

13. Prove that $\forall x$: $x + 0 = x$.

14. Prove that $\forall x, y$ there is a unique number z such that $x = y + z$ (z is the difference of the numbers x and y, i.e. $z = x - y$).

15. Prove that $\forall x$: $x \cdot 1 = x$.

16. Prove that $\forall x \neq 0$ $\exists x'$: $xx' = 1$.

17. Prove that $\forall x$ and $\forall y \neq 0$ there is a unique number z such that $x = yz$ (z is the quotient of the division of x by y, i.e. $z = x/y$).

18. Prove that $\forall x, y, z$: $(x + y)z = xz + yz$.

19. Prove that if $x > y$, then $\forall z$: $x + z > y + z$.

20. Prove that if $x > y$, then $\forall z > 0$: $xz > yz$.

21. Prove the validity of the following inequalities (a) $|x + y| \leqslant |x| + |y|$, (b) $|x - y| \geqslant |x| - |y|$.

22. Assume that X and Y are nonempty bounded sets of real numbers and T is the set of various sums $x + y$, where $x \in X$ and $y \in Y$. Prove that the set T is bounded and that (a) $\sup T = \sup X + \sup Y$, (b) $\inf T = \inf X + \inf Y$.

23. Assume that X and Y are nonempty bounded sets of nonnegative real numbers and B is the set of various products xy, where $x \in X$ and $y \in Y$. Prove that the set B is bounded and that (a) $\sup B = \sup X \cdot \sup Y$, (b) $\inf B = \inf X \cdot \inf Y$.

24. Calculate the first three significant digits of the sums (a) $\frac{1}{7} + \sqrt{3}$, (b) $\sqrt{3} + \sqrt{7}$.

25. Find the first three decimal digits of the products (a) $\frac{1}{7}\sqrt{3}$, (b) $\sqrt{3} \cdot \sqrt{7}$.

26. Assume that A and B are nonempty sets of real numbers in which every number from A is smaller than any number from B, and for every $\varepsilon > 0$ there are $x \in A$ and $y \in B$ such that $y - x < \varepsilon$. Prove that $\sup A = \inf B$.

1.4. Induction

I. Fundamental Concepts

To prove that a statement is true for any natural number n beginning with n_0, it is sufficient to prove that

(a) the statement is true for $n = n_0$,

(b) if the statement is true for some natural number $k \geqslant n_0$, it is also true for the next natural number $k + 1$.

This method of proof is known as *mathematical induction.*

II. Control Questions

1. What is the gist of the method of induction?
2. Use induction to prove that $\forall n \in \mathbf{N}$: $n \leqslant 2^{n-1}$.

III. Worked Problems

1. Prove that $\forall n \in \mathbf{N}$ and $\forall x > -1$ the inequality

$$(1 + x)^n \geqslant 1 + nx \quad \textbf{(Bernoulli's inequality)} \qquad (1)$$

holds true.

△ We shall use induction to prove inequality (1). If $n = 1$, then inequality (1) holds true since it becomes a true equality. We assume that relation (1) holds true for a natural number k and $\forall x > -1$:

$$(1 + x)^k \geqslant 1 + kx, \qquad (2)$$

Since $x > -1$, it follows that $1 + x > 0$. We multiply inequality (2) by the positive number $+1 + x$:

$$(1 + x)^{k+1} \geqslant 1 + kx + x + kx^2.$$

Removing the nonnegative term kx^2 on the right-hand side, we obtain an inequality

$$(1 + x)^{k+1} \geqslant 1 + (k + 1) x.$$

We have proved that inequality (1) is valid for the natural number $k + 1$ and $\forall x > -1$. We have thus proved that relation (1) is valid $\forall n \in \mathbf{N}$ and $\forall x > -1$. ▲

2. Prove that for any n positive numbers $y_1, y_2, \ldots, y_n$ which satisfy the condition

$$y_1 y_2 \ldots y_n = 1, \tag{3}$$

there holds a relation

$$y_1 + y_2 + \ldots + y_n \geqslant n. \tag{4}$$

△ It follows from condition (3) for $n = 1$ that $y_1 = 1$. Therefore relation (4) is satisfied.

Assume that relation (4) follows from condition (3) for $n = k$ and that $k + 1$ positive numbers $y_1, y_2, \ldots, y_k, y_{k+1}$ satisfy condition (3). We shall prove that relation (4) is satisfied for them. If all these numbers are equal to unity, then their sum is equal to $k + 1$ and relation (4) holds true. Now if there is at least one number different from unity among the indicated numbers, then there is necessarily one more number which is not equal to unity, and if one number is larger than 1, then the other is smaller than 1. Without violating the generality, we assume that $y_k > 1$ and $y_{k+1} < 1$. The product of k numbers $y_1, y_2, \ldots, y_{k-1}, y_k y_{k+1}$ is equal to 1 by virtue of condition (3). Therefore, on inductional assumption

$$y_1 + y_2 + \ldots + y_{k-1} + y_k y_{k+1} \geqslant k.$$

Hence we obtain

$$y_1 + y_2 + \ldots + y_{k+1} + y_k y_{k+1} \geqslant k + y_k + y_{k+1},$$

or

$$y_1 + y_2 + \ldots + y_{k+1} \geqslant k + 1 + y_k + y_{k+1} - y_k y_{k+1} - 1 = k + 1 + (1 - y_{k+1})(y_k - 1) \geqslant k + 1,$$

i.e. relation (4) is satisfied for $n = k + 1$. Thus for any n positive numbers, which satisfy condition (3), relation (4) holds true.

IV. Problems and Exercises for Independent Work

Using induction, prove that $\forall n \in \mathrm{N}$ the following equalities hold true:

27. $1 + 2 + 3 + \ldots + = 0.5n(n + 1)$.

28. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6} n(n+1)(2n+1)$.

29. $1^3 + 2^3 + 3^3 + \ldots + n^3 = 0.25n^2(n + 1)^2$.

Use induction to prove the validity of the following inequalities:

30. $\frac{1}{2} \cdot \frac{3}{4} \ldots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

31. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n} \quad (n \geqslant 2)$.

32. $n^{n+1} > (n+1)^n \quad (n \geqslant 3)$.

33. $\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n}$ for $x_k \geqslant 0$, $k = 1, \ldots, n$ (the geometric mean of n nonnegative numbers does not exceed their arithmetic mean).

34. $\frac{\sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}}{n} \leqslant 0.5(1 + \sqrt{4a+1})$, $\forall a > 0$.

Chapter 2

Limit of a Sequence

2.1. Bounded and Unbounded Sequences. Limit of a Sequence

I. Fundamental Concepts and Theorems

If every natural number n is associated with a number x_n, then we say that *a number sequence* (or simply a *sequence*) $x_1, x_2, x_3, \ldots, x_n, \ldots$ is defined. It is denoted by the symbol $\{x_n\}$ or (x_n). The number x_n is a *term* (*element*) of the sequence and n is the *number* of the term.

The sequences $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{x_n y_n\}$, $\{x_n/y_n\}$ are the *sum*, the *difference*, the *product* and the *quotient* of two sequences $\{x_n\}$ and $\{y_n\}$ respectively (for the quotient $y_n \neq 0$).

Definition. *The sequence* $\{x_n\}$ *is said to be* ***bounded*** *if* $\exists M > 0$ *such that* $\forall n$: $|x_n| \leqslant M$.

From the point of view of geometry, this means that all terms of the sequence are in a certain neighbourhood (M-neighbourhood) of the point $x = 0$.

Definition. *The sequence* $\{x_n\}$ *is said to be* ***unbounded*** *if* $\forall M > 0$ $\exists n$: $|x_n| > M$.

Definition. *The number a is a* ***limit of the sequence*** $\{x_n\}$ *if* $\forall \varepsilon > 0$ $\exists N \in \mathrm{N}$ *such that* $\forall n > N$: $|x_n - a| < \varepsilon$. The designation is $\lim\limits_{n\to\infty} x_n = a$.

From the point of view of geometry, this means that any ε-neighbourhood of the point a contains all terms of the sequence beginning with a certain number (dependent, in general, on ε), or, what is the same, outside of any ε-neighbourhood of the point a there is only a finite number of terms of the sequence.

A sequence which has a limit is called a *convergent* sequence and a sequence which has no limit is called a *divergent* sequence.

Theorem 1. *A convergent sequence has only one limit.*

Theorem 2 (the necessary condition for convergence of a sequence). *A convergent sequence is bounded.*

II. Control Questions and Assignments

1. Formulate the definitions of (a) a sequence, (b) a bounded and unbounded sequence, (c) a limit of a sequence. Give a geometric interpretation of these definitions.

2. Find whether the definition of a limit of a sequence is equivalent to the following definition: $\lim\limits_{n\to\infty} x_n = a$ if $\forall \varepsilon > 0$ $\exists$ a positive number K (not necessarily natural) such that $\forall n > K$: $|x_n - a| < \varepsilon$.

3. Give an example showing that the number N appearing in the definition of a limit of a sequence depends, in general, on ε.

4. Let the sequence $\{x_n\}$ and the number a satisfy the following condition: $\exists N$ such that $\forall \varepsilon > 0$ and $\forall n > N$: $|x_n - a| < \varepsilon$. Does every sequence con-

verging to a satisfy this condition? What is the geometric interpretation of this condition?

5. Let $\lim\limits_{n\to\infty} x_n = a$.

(a) Can all the terms of a sequence be positive (negative) if $a = 0$?

(b) Can a sequence have infinitely many negative (equal to zero) terms if $a > 0$, $a \neq 0$?

(c) Prove that $\lim\limits_{n\to\infty} x_{n+1} = a$, $\lim\limits_{n\to\infty} x_{n+2} = a$.

(d) Prove that $\{x_n\}$ is bounded.

6. Let infinitely many terms of the sequence $\{x_n\}$ lie in a certain neighbourhood of the point a. Does it follow from this condition that (a) $\lim\limits_{n\to\infty} x_n = a$, (b) none of the points which lie outside of this neighbourhood is a limit of the sequence $\{x_n\}$ and (c) $\{x_n\}$ is bounded?

7. Let infinitely many terms of the sequence $\{x_n\}$ lie in any neighbourhood of the point a. Does it follow that (a) $\lim\limits_{n\to\infty} x_n = a$ and (b) $\{x_n\}$ is bounded?

8. What sequence is (a) convergent, (b) divergent?

9. Let the sequence $\{x_n\}$ be bounded (unbounded). Does it follow from this condition that it is convergent (divergent)?

10. Let a sequence be convergent. Is a sequence, which results from the original one, convergent if:

(a) a finite number of terms are removed and the remaining terms are enumerated anew in the order of their appearance?

(b) a finite number of terms are added and all the terms are enumerated anew in the order of their appearance?

(c) a finite number of terms are changed arbitrarily?

11. Prove that a convergent sequence has only one limit.

12. Formulate the necessary condition for convergence of a sequence.

III. Worked Problems

1. Using the "ε-N" language, show that the number a is not a limit of the sequence $\{x_n\}$ ($a \neq \lim\limits_{n\to 0} x_n$) and give a geometric interpretation of this definition.

△ In accordance with the definition of a limit of a sequence, $a = \lim\limits_{n\to 0} x_n$ if $\forall \varepsilon > 0$ $\exists N$ such that $\forall n > N$:

$|x_n - a| < \varepsilon$. Using the rule of constructing negations, we find that $a \neq \lim_{n\to\infty} x_n$ if $\exists \varepsilon > 0$ such that $\forall N\ \exists n > N$: $|x_n - a| \geqslant \varepsilon$.

It can be written in more detail as follows: $a \neq \lim_{n\to\infty} x_n$ if $\exists \varepsilon > 0$ such that

$$\begin{array}{lll} \exists n_1 > 1: & |x_{n_1} - a| \geqslant \varepsilon & \text{for } N = 1, \\ \exists n_2 > 2: & |x_{n_2} - a| \geqslant \varepsilon & \text{for } N = 2, \\ \dots & \dots & \dots \\ \exists n_{100} > 100: & |x_{n_{100}} - a| \geqslant \varepsilon & \text{for } N = 100. \\ \dots & \dots & \dots \end{array}$$

Thus $a \neq \lim_{n\to\infty} x_n$ if there are $\varepsilon > 0$ and a sequence of numbers $\{n_N\}$ $(N = 1, 2, 3, \dots)$ such that $n_N > N$ and $|x_{n_N} - a| \geqslant \varepsilon$.

The geometric interpretation of this definition is: $a \neq \lim_{n\to\infty} x_n$ if there is a ε-neighbourhood of the point a outside of which there are infinitely many terms of the sequence. ▲

2. Prove that the sequence $x_n = 2^{n(-1)^n}$ is unbounded.

△ By virtue of the definition of an unbounded sequence, we must show that $\forall M > 0$, $\exists n \in \mathbf{N}$ for which $|x_n| > M$. We take an arbitrary $M > 0$ and any even number n which satisfies the inequality $n > \log_2 M$. For such an n we have

$$x_n = 2^n > 2^{\log_2 M} = M,$$

and this is what we wished to prove. ▲

3. Using the definition of a limit of a sequence, prove that $\lim_{n\to\infty} \frac{5\cdot 3^n}{3^n - 2} = 5$.

△ We specify an arbitrary $\varepsilon > 0$ and consider the modulus of the difference of the nth term of the sequence and the number 5:

$$\left| \frac{5\cdot 3^n}{3^n - 2} - 5 \right| = \frac{10}{3^n - 2}.$$

In accordance with the definition of a limit of a sequence, we must indicate a number N such that $\forall n > N$ the

inequality

$$\frac{10}{3^n-2}<\varepsilon \tag{1}$$

is satisfied. To find the number N, we solve inequality (1) for n. We obtain

$$n>\log_3\left(\frac{10}{\varepsilon}+2\right). \tag{2}$$

It follows from inequality (2) that we can take as N the integral part of the number $\log_3\left(\frac{10}{\varepsilon}+2\right)$:

$$N=\left[\log_3\left(\frac{10}{\varepsilon}+2\right)\right].$$

Indeed, if $n>N$, then

$$n\geqslant\left[\log_3\left(\frac{10}{\varepsilon}+2\right)\right]+1>\log_3\left(\frac{10}{\varepsilon}+2\right),$$

i.e. inequality (2) holds true, and this means that $\forall n>N$ the inequality (1) also holds true. $\Big($Note that for $\varepsilon>10$ we have $N=\left[\log_3\left(\frac{10}{\varepsilon}+2\right)\right]=0$ and therefore inequality (1) is valid $\forall n\in\mathbb{N}.\Big)$

Thus, for an arbitrary $\varepsilon>0$ we have indicated the number $N=\left[\log_3\left(\frac{10}{\varepsilon}+2\right)\right]$, such that $\forall n>N$ the inequality

$$\left|\frac{5\cdot 3^n}{3^n-2}-5\right|<\varepsilon$$

holds true. And this means, according to the definition of a limit of a sequence, that

$$\lim_{n\to\infty}\frac{5\cdot 3^n}{3^n-2}=5. \quad \blacktriangle$$

4. Using the definition of a limit of a sequence, prove that $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n!}}=0$.

$\triangle$ We specify an arbitrary $\varepsilon>0$. We must indicate a number N such that $\forall n>N$ the inequality

$$1/\sqrt{n!}<\varepsilon \tag{3}$$

holds true. We shall not try to find the least number N, beginning with which inequality (3) holds true, but shall indicate a larger number "by excess" and solve a simpler inequality

$$2/n < \varepsilon. \tag{4}$$

Since $\forall n \in \mathrm{N}$: $n! > n\,(n/4)$ (prove this), it follows that $\forall n \in \mathrm{N}$ the inequality

$$1/\sqrt{n!} < 2/n \tag{5}$$

holds true and therefore inequality (3) is a consequence of inequality (4). Solving inequality (4) for n, we obtain

$$n > 2/\varepsilon. \tag{6}$$

We set $N = [2/\varepsilon]$. If $n > N$, then $n \geqslant [2/\varepsilon] + 1 > 2/\varepsilon$, i.e. inequality (6) is satisfied and, consequently, inequalities (4) and (3) are also satisfied. Thus $\forall \varepsilon > 0\ \exists N$ $(N = [2/\varepsilon])$ such that $\forall n > N$: $1/\sqrt{n!} < \varepsilon$. We have thus proved that $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n!}} = 0$. ▲

The examples we have considered show the way to prove that $\lim\limits_{n\to\infty} x_n = a$ using the definition of a limit of a sequence. It is necessary to form an expression $|x_n - a|$ and choose (if it is expedient) a sequence $\{y_n\}$ such that, first, $\forall n\ |x_n - a| \leqslant y_n$ and, second, for an arbitrary ε it is easy to solve the inequality

$$y_n < \varepsilon \tag{7}$$

for n. Assume that the solution of inequality (7) has the form

$$n > f(\varepsilon),$$

where $f(\varepsilon) > 0$. Then we can take $[f(\varepsilon)]$ as N (if it turns out that $[f(\varepsilon)] = 0$, then inequality (7) is valid $\forall n$). Thus $\forall n > N = [f(\varepsilon)]$ there holds an inequality $|x_n - a| < \varepsilon$ and this means, according to the definition of a limit of a sequence, that $\lim\limits_{n\to\infty} x_n = a$.

5. It is known that $\lim\limits_{n\to\infty} x_n = 0$ and $x_n \geqslant 0\ \forall n$. Prove that $\lim\limits_{n\to\infty} x_n^\alpha = 0$ for $\alpha > 0$.

△ By the hypothesis, $\lim\limits_{n\to\infty} x_n = 0$, i.e. $\forall \varepsilon_1 > 0\ \exists N_1$ such that $\forall n > N_1$ the inequality

$$|x_n| < \varepsilon_1 \tag{8}$$

holds true. We have to prove that $\forall\varepsilon > 0\ \exists N$ such that $\forall n > N$: $|x_n^\alpha| < \varepsilon$ or, what is the same,

$$|x_n| < \varepsilon^{1/\alpha}. \tag{9}$$

We specify an arbitrary $\varepsilon > 0$ and set $\varepsilon_1 = \varepsilon^{1/\alpha}$ ($\varepsilon_1 > 0$). For this ε_1 $\exists N_1$ such that $\forall n > N_1$ inequality (8) holds true, i.e. $|x_n| < \varepsilon^{1/\alpha}$. Thus $\forall n > N = N_1$ inequality (9) holds true. We have thus proved that $\lim\limits_{n\to\infty} x_n^\alpha = 0$. ▲

IV. Problems and Exercises for Independent Work

1. Find out whether each of the following sequences is bounded: (a) $x_n = (-1)^n \frac{1}{n}$, (b) $x_n = 2n$, (c) $x_n = \ln n$, (d) $x_n = \sin n$, (e) $\{x_n\} = 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots$. Substantiate the answer.

2. Using the definition of a limit of a sequence, prove that

(a) $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$, (b) $\lim\limits_{n\to\infty} \frac{2n}{n+3} = 2$,

(c) $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$, (d) $\lim\limits_{n\to\infty} \log_n 2 = 0$,

(e) $\lim\limits_{n\to\infty} \frac{1}{n^3+2n+1} = 0$, (f) $\lim\limits_{n\to\infty} (0.8)^n = 0$,

(g) $\lim\limits_{n\to\infty} \frac{2^n+5\cdot 6^n}{3^n+6^n} = 5$, (h) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2}\sin n^2}{n+1} = 0$.

3. It is known that $\lim\limits_{n\to\infty} x_n = a$. Prove that

(a) $\lim\limits_{n\to\infty} (x_{n+1} - x_n) = 0$, (b) $\lim\limits_{n\to\infty} |x_n| = |a|$, (c) $\lim\limits_{n\to\infty} x_n^2 = a^2$.

4. Let $\lim\limits_{n\to\infty} |x_n| = |a|$. Does it follow that $\lim\limits_{n\to\infty} x_n = a$?

5. Prove that the sequence $\{x_n\}$ is divergent when (a) $x_n = n$, (b) $x_n = \ln n$, (c) $x_n = n^{(-1)^n}$.

6. Assume that the sequence $\{x_n\}$ is convergent and $M = \sup\{x_n\}$, $m = \inf\{x_n\}$. Prove that either $\exists n$ such that $x_n = M$, or $\exists k$ such that $x_k = m$, or $\exists n$ and k such that $x_n = M$, $x_k = m$. Give examples of sequences of these three kinds.

2.2. Infinitely Small and Infinitely Large Sequences

I. Fundamental Concepts and Theorems

Definition. *The sequence* $\{x_n\}$ *is* ***infinitely small (infinitesimal)*** *if* $\lim_{n\to\infty} x_n = 0$.

Definition. *The sequence* $\{x_n\}$ *is* ***infinitely large*** *if* $\forall A > 0 \ \exists N$ *such that* $\forall n > N$: $|x_n| > A$.

From the point of view of geometry, this means that in any (infinitely large) neighbourhood of zero there are only a finite number of terms of the sequence and infinitely many terms outside of it.

If the sequence $\{x_n\}$ is infinitely large, then we write $\lim_{n\to\infty} x_n = \infty$, and if, beginning with a certain number, all terms of an infinitely large sequence are positive (negative), then we write $\lim_{n\to\infty} x_n = +\infty \ (-\infty)$. Note that an infinitely large sequence is not convergent and the symbolic notation $\lim_{n\to\infty} x_n = +\infty \ (-\infty)$ only means that the sequence $\{x_n\}$ is infinitely large and does not mean at all that it has a limit.

Every infinitely large sequence is unbounded since there is a term of the sequence (even all terms beginning with a certain number) outside of any neighbourhood of zero. The converse is not true: an unbounded sequence may not be infinitely large.

Theorem 3. *The algebraic sum of a finite number of infinitesimal sequences is an infinitesimal sequence.*

Theorem 4. *The product of an infinitesimal sequence by a bounded sequence is an infinitesimal sequence.*

Corollary. *The product of a finite number of infinitesimals is an infinitesimal.*

Theorem 5. *If the sequence* $\{x_n\}$ *is infinitely large, then, beginning with a certain number* n, *the sequence* $\{1/x_n\}$ *which is infinitesimal is defined. If the sequence* $\{x_n\}$ *is infinitesimal and* $x_n \neq 0 \ \forall n$, *then the sequence* $\{1/x_n\}$ *is infinitely large.*

II. Control Questions and Assignments

1. Formulate the definitions of: (a) an infinitesimal sequence, (b) an infinitely large sequence. Give a geometric interpretation of these definitions.

2. Use the "ε-N" language to formulate the negation of the fact that a sequence is (a) infinitesimal, (b) infinitely large. Give a geometric interpretation of these negations.

3. Give a definition corresponding to the symbolic notation $\lim\limits_{n\to\infty} x_n = -\infty$.

4. Let an infinite number of terms of a sequence be (a) in any neighbourhood of zero, (b) outside of any neighbourhood of zero. Does it follow from condition (a) (condition (b)) that the sequence is infinitesimal? infinitely large? bounded? unbounded? Does it follow from condition (a) (condition (b)) that the sequence is not infinitesimal? not infinitely large?

5. It is known that the sequence $\{x_n\}$ is (a) infinitesimal, (b) infinitely large. Does it follow (under the condition that $x_n \neq 0\ \forall n$) that the sequence $\{1/x_n\}$ is (a) infinitely large, (b) infinitesimal?

6. (a) Is an infinitesimal sequence bounded?

(b) Is an infinitely large sequence unbounded? convergent?

(c) Is any unbounded sequence infinitely large?

7. It is known that $y_n \neq 0\ \forall n$ and: (a) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$, (b) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = \infty$. Can the sequence $\{x_n/y_n\}$ be infinitely large? infinitesimal? convergent? divergent, but not infinitely large? Give examples.

8. Prove that the sum of two infinitesimal sequences is infinitesimal. Is an analogous statement for an infinitely large sequence true? Substantiate the answer.

9. Prove that if $\lim\limits_{n\to\infty} x_n = \infty$, then, beginning with a certain number n, the sequence $\{1/x_n\}$ is defined, with $\lim\limits_{n\to\infty} (1/x_n) = 0$.

10. Let $\{x_n + y_n\}$ be an infinitesimal sequence. Does it follow that $\{x_n\}$ and $\{y_n\}$ are infinitesimal sequences? Substantiate the answer.

11. Let $\{x_n y_n\}$ be an infinitesimal sequence. Does it follow that at least one of the sequences $\{x_n\}$ and $\{y_n\}$ is infinitesimal? Substantiate the answer.

12. Prove that if $x_n \geqslant y_n$ and $\lim\limits_{n\to\infty} y_n = +\infty$, then $\lim\limits_{n\to\infty} x_n = +\infty$.

III. Worked Problems

1. Use the "ε-N" language to formulate the negation of the fact that a sequence is infinitely large. Give a geometric interpretation of the negation.

$\triangle$ In accordance with the definition of an infinitely large sequence, $\lim\limits_{n\to\infty} x_n = \infty$ if $\forall A > 0$ $\exists N$ such that $\forall n > N$: $|x_n| > A$. Using the rule of constructing negations, we find that the sequence $\{x_n\}$ is not infinitely large if $\exists A > 0$ such that $\forall N$ $\exists n > N$: $|x_n| \leqslant A$. From the point of view of geometry this means that there is a neighbourhood of zero (A neighbourhood) which contains infinitely many terms of the sequence. $\blacktriangle$

2. Prove that the sequence $\{a^n\}$ is (a) infinitely large for $|a| > 1$, (b) infinitesimal for $|a| < 1$.

$\triangle$ Let $|a| > 1$. We shall prove that the sequence $\{a^n\}$ satisfies the definition of an infinitely large sequence, i.e. $\forall A > 0$ $\exists N$ such that $\forall n > N$ there holds an inequality

$$|a|^n > A. \tag{1}$$

We specify an arbitrary $A > 0$. To find the number N, we solve inequality (1) for n. We get

$$n > \log_{|a|} A. \tag{2}$$

We set $N = [\log_{|a|} A]$. Then $\forall n > N$ inequality (2) holds true, and, hence, inequality (1) also holds true. Thus $\forall A > 0$ $\exists N = [\log_{|a|} A]$ such that $\forall n > N$: $|a|^n > A$. And this is what we had to prove.

(b) Let $|a| < 1$. If $a = 0$, then $a^n = 0$ $\forall n$ and, consequently, $\{a^n\}$ is an infinitesimal sequence. Let $a \neq 0$. Then $a^n = ((1/a)^n)^{-1}$. Since $|1/a| > 1$, the sequence $\{(1/a)^n\}$ is infinitely large and the sequence $((1/a)^n)^{-1}$ is infinitesimal by virtue of Theorem 5. Thus the sequence $\{a^n\}$ is infinitesimal for $|a| < 1$. $\blacktriangle$

3. Let $x_n = n^{(-1)^n}$. Prove that the sequence $\{x_n\}$ (a) is unbounded, (b) is not infinitely large.

$\triangle$ (a) We shall prove that $\{x_n\}$ satisfies the definition of an unbounded sequence. Indeed, $\forall M > 0$ the term of the sequence with the number $n = 2([M] + 1)$ is equal to n and exceeds M, and this means, by definition, that $\{x_n\}$ is an unbounded sequence.

(b) We shall prove that the sequence $\{x_n\}$ is not infinitely large. Indeed, the interval $(-2, 2)$ evidently

includes all terms of the sequence with odd numbers and, hence, the interval contains infinitely many terms of the sequence $\{x_n\}$. It follows that $\{x_n\}$ is not an infinitely large sequence. ▲

4. Assume that $\{x_n\}$ is a convergent sequence and $\{y_n\}$ is an infinitely large sequence. Prove that the sequence $\{x_n + y_n\}$ is infinitely large.

△ We shall prove that the sequence $\{x_n + y_n\}$ satisfies the definition of an infinitely large sequence, i.e. $\forall A > 0 \quad \exists N$ such that $\forall n > N$: $|x_n + y_n| > A$. Since $\{x_n\}$ is convergent, it is bounded, i.e. $\exists M > 0$ such that $\forall n$ there holds an inequality

$$|x_n| < M. \tag{3}$$

We specify an arbitrary $A > 0$. Since $\{y_n\}$ is an infinitely large sequence, it follows that for the number $A + M$ $\exists N$ such that $\forall n > N$ we have

$$|y_n| > A + M. \tag{4}$$

From (3) and (4) we find that $\forall n > N$ there holds an inequality

$$|x_n + y_n| \geqslant |y_n| - |x_n| > A + M - M = A,$$

and this is what we had to prove. ▲

IV. Problems and Exercises for Independent Work

7. It is known that in a neighbourhood of zero there are (a) a finite number of terms of a sequence, (b) an infinite number of terms of a sequence. Does it follow that in each of these cases the sequence is bounded? infinitesimal? infinitely large?

8. It is known that the sequence $\{x_n\}$ is convergent and the sequence $\{y_n\}$ is infinitely large. Can the sequence $\{x_n y_n\}$ be (a) convergent, (b) divergent but bounded, (c) infinitely large, (d) infinitesimal? Answer the questions using the sequences $\{n\}$, $\left\{\frac{n-1}{n}\right\}$, $\left\{\frac{1}{n}\right\}$, $\left\{\frac{(-1)^n}{n}\right\}$, $\left\{\frac{1}{n^2}\right\}$ as examples.

9. Give examples of sequences $\{x_n\}$ and $\{y_n\}$ for which $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{n\to\infty} y_n = \infty$ and their product $\{x_n y_n\}$

is (a) a convergent sequence, (b) a divergent but bounded sequence, (c) an infinitesimal sequence, (e) an infinitely large sequence.

10. Prove that the following specified sequences are infinitesimal: (a) $x_n = n^k$ $(k < 0)$, (b) $x_n = (-1)^n \cdot 0.999^n$, (c) $x_n = \frac{1}{n!}$, (d) $x_n = \frac{n}{2n^3+1}$.

11. Prove that the following specified sequences are infinitely large: (a) $x_n = n^k$ $(k > 0)$, (b) $x_n = n^{(-1)^n}$, (c) $x_n = 2^{\sqrt{n}}$, (d) $x_n = \log_2(\log_2 n)$ $(n \geqslant 2)$.

12. Prove that any infinitely large sequence is unbounded.

13. Prove that the sequence $\{(1 + (-1)^n)\, n\}$ is unbounded but not infinitely large.

14. Prove that if $\lim\limits_{n\to\infty} x_n = +\infty$ $(-\infty)$, then the sequence $\{x_n\}$ attains its greatest lower (least upper) bound.

15. Find the smallest term of the sequence $\{x_n\}$ if (a) $x_n = n^2 - 9n - 100$, (b) $x_n = n + \frac{100}{n}$.

2.3. Properties of Convergent Sequences

I. Fundamental Concepts and Theorems

Theorem 6. *Let* $\lim\limits_{n\to\infty} x_n = a$, $\lim\limits_{n\to\infty} y_n = b$. *Then*

(a) $\lim\limits_{n\to\infty} (x_n + y_n) = a + b$,

(b) $\lim\limits_{n\to\infty} (x_n y_n) = ab$,

(c) *if* $b \neq 0$, *then, beginning with a certain number a sequence* $\{x_n/y_n\}$ *is defined (i.e.* $\exists N$ *such that* $\forall n \geqslant N$: $y_n \neq 0$) *and* $\lim\limits_{n\to\infty} (x_n/y_n) = a/b$.

If $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$, then $\lim\limits_{n\to\infty} (x_n/y_n)$ is an *indeterminate form* 0/0. The indeterminate forms ∞/∞, $0\cdot\infty$, $\infty - \infty$ are defined by analogy. It is clear that Theorem 6 is inapplicable for these limits.

Theorem 7. *If* $\lim\limits_{n\to\infty} x_n = a$ *and, beginning with a certain number,* $x_n \geqslant b$ $(x_n \leqslant b)$, *then* $a \geqslant b$ $(a \leqslant b)$.

Theorem 8 (theorem on three sequences). *If* $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = a$ *and, beginning with a certain number, the inequalities* $x_n \leqslant z_n \leqslant y_n$ *hold true, then* $\lim_{n\to\infty} z_n = a$.

II. Control Questions and Assignments

1. Give a definition of a convergent sequence.
2. Use the "ε-N" language to give the definition of a divergent sequence and give a geometric interpretation of this definition.
3. Formulate Theorems 6-8.
4. Let the sequence $\{x_n\}$ converge and the sequence $\{y_n\}$ diverge. Prove that the sequence $\{x_n + y_n\}$ diverges, the sequence $\{cx_n\}$ converges, the sequence $\{cy_n\}$ diverges for $c \neq 0$. Use examples to show that the sequence $\{x_n y_n\}$ may (a) converge, (b) diverge.
5. Let the sequence $\{x_n + y_n\}$ be convergent. Does it follow that the sequences $\{x_n\}$ and $\{y_n\}$ are convergent?
6. Let

$$\lim_{n\to\infty} x_n = a. \qquad (*)$$

Prove that x_n can be represented in the form

$$x_n = a + \alpha_n, \qquad (**)$$

where $\{\alpha_n\}$ is an infinitesimal sequence. Prove the converse, i.e. that (*) follows from (**).

7. Prove Theorem 6.
8. Let $\lim_{n\to\infty} x_n = a$ and $x_n > b$ $\forall n$. Does it follow that (a) $a > b$, (b) $a \geqslant b$?

III. Worked Problems

1. Assume that $\lim_{n\to\infty} y_n = b \neq 0$ and the sequence $\{z_n\}$ is divergent. Prove that $\{z_n y_n\}$ diverges.

△ We designate $x_n = z_n y_n$ and prove the divergence of the sequence $\{x_n\}$ by the method of indirect proof. We assume that $\{x_n\}$ converges. Since $\lim_{n\to\infty} y_n = b \neq 0$ by the hypothesis, it follows, according to Theorem 6, that the sequence $\{x_n/y_n\} = \{z_n\}$ is defined beginning with a certain number and is convergent. But this con-

tradicts the hypothesis. Consequently, $\{x_n\}$ is divergent. ▲

2. Prove that the sequence $\{\sin n\}$ is divergent.

△ We shall use the method of indirect proof. Let $\lim\limits_{n\to\infty} \sin n = a$. Then $\lim\limits_{n\to\infty} \sin(n+2) = a$, whence

$$\lim_{n\to\infty} (\sin(n+2) - \sin n) = 0. \tag{1}$$

Since $\sin(n+2) - \sin n = 2 \sin 1 \cos(n+1)$, we can take relation (1) into account and obtain

$$\lim_{n\to\infty} \cos(n+1) = 0. \tag{2}$$

From the relation $\cos(n+1) = \cos n \cos 1 - \sin n \sin 1$ we find that $\sin n = \frac{1}{\sin 1}(\cos n \cos 1 - \cos(n+1))$. From this it follows, by virtue of (2), that $\lim\limits_{n\to\infty} \sin n = 0$. Thus we get $\lim\limits_{n\to\infty} \cos n = \lim\limits_{n\to\infty} \sin n = 0$, and this contradicts the equality $\cos^2 n + \sin^2 n = 1$. Consequently, $\{\sin n\}$ diverges. ▲

3. Find the following limits:

$$\text{(a)} \lim_{n\to\infty} \frac{10}{n^2+1}, \quad \text{(b)} \lim_{n\to\infty} \frac{n^2-n}{n-\sqrt{n}}, \quad \text{(c)} \lim_{n\to\infty} \frac{5\cdot 3^n}{3^n-2}.$$

△ Note that each of these limits is an indeterminate form of type ∞/∞. We have

$$\text{(a)} \lim_{n\to\infty} \frac{10n}{n^2+1} = \lim_{n\to\infty} \frac{10}{n+\frac{1}{n}} = 0$$

since $\left\{n+\frac{1}{n}\right\}$ is an infinitely large sequence,

$$\text{(b)} \lim_{n\to\infty} \frac{n^2-n}{n-\sqrt{n}} = \lim_{n\to\infty} \frac{(n-\sqrt{n})(n+\sqrt{n})}{n-\sqrt{n}}$$

$$= \lim_{n\to\infty} (n+\sqrt{n}) = +\infty,$$

$$\text{(c)} \lim_{n\to\infty} \frac{5\cdot 3^n}{3^n-2} = \lim_{n\to\infty} \frac{5}{1-\frac{2}{3^n}}$$

$$= 5\,\frac{1}{\lim\limits_{n\to\infty}\left(1-\frac{2}{3^n}\right)} = 5. \ ▲$$

4. Find the limit $\lim\limits_{n\to\infty}(\sqrt{n^2+n}-n)$.

△ Note that this limit is an indeterminate form of type $\infty-\infty$. We have

$$\lim_{n\to\infty}(\sqrt{n^2+n}-n)=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{\lim\limits_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}\right)+1}=\frac{1}{2}. \quad ▲$$

5. Calculate $\lim\limits_{n\to\infty}\frac{\sqrt{n}\cos n}{n+1}$.

△ The sequence $\{\cos n\}$ is bounded and the sequence $\left\{\frac{\sqrt{n}}{n+1}\right\}$ is infinitesimal since

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}}{1+\frac{1}{n}}=\frac{\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}=0.$$

From this, according to Theorem 4, it follows that the product of these sequences is an infinitesimal sequence, i.e. $\lim\limits_{n\to\infty}\frac{\sqrt{n}\cos n}{n+1}=0$. ▲

6. Find the limit $\lim\limits_{n\to\infty}\frac{1^4+2^4+3^4+\ldots+n^4}{n^5}$.

△ We designate $S_n=1^4+2^4+3^4+\ldots+n^4$. We shall seek S_n in the form

$$S_n=An^5+Bn^4+Cn^3+Dn^2+En+F,$$

and then

$$S_{n+1}-S_n=A\left[(n+1)^5-n^5\right]+B\left[(n+1)^4-n^4\right]$$
$$+C\left[(n+1)^3-n^3\right]+D\left[(n+1)^2-n^2\right]$$
$$+E\left[(n+1)-n\right].$$

From this we have the following relation for any natural n:

$$(n+1)^4=5An^4+(10A+4B)n^3$$
$$+(10A+6B+3C)n^2+(5A+4B+3C+2D)n$$
$$+A+B+C+D+E.$$

Equating the coefficients in the same power of n on the left-hand and right-hand sides of the relation, we obtain

$$\begin{aligned} 5A \qquad\qquad\qquad\qquad &= 1, \\ 10A + 4B \qquad\qquad\qquad &= 4, \\ 10A + 6B + 3C \qquad\quad &= 6, \\ 5A + 4B + 3C + 2D \quad &= 4, \\ A + B + C + D + E &= 1. \end{aligned}$$

Hence $A=1/5$, $B=1/2$, $C=1/3$, $D=0$, $E=-1/30$. Thus, for any n, we have $S_n=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n+F$. Setting $n=1$, we get $1=\frac{1}{5}+\frac{1}{2}+\frac{1}{3}-\frac{1}{30}+F$, whence $F=0$. Consequently,

$$S_n=1^4+2^4+3^4+\ldots+n^4=\frac{6n^5+15n^4+10n^3-n}{30}.$$

Thus we have

$$\lim_{n\to\infty}\frac{1^4+2^4+3^4+\ldots+n^4}{n^5}$$

$$=\lim_{n\to\infty}\left(\frac{1}{5}+\frac{1}{2n}+\frac{1}{3n^2}-\frac{1}{30n^4}\right)=\frac{1}{5}. \ \blacktriangle$$

IV. Problems and Exercises for Independent Work

16. (a) It is known that the sequence $\{x_n\}$ is convergent and $\{y_n\}$ is divergent. Can the sequence $\{x_ny_n\}$ be convergent? divergent?

(b) The sequences $\{x_n\}$ and $\{y_n\}$ are known to be divergent. Can the sequences $\{x_n+y_n\}$ and $\{x_ny_n\}$ be convergent? divergent?

Answer the questions using the sequences $\left\{\frac{n+1}{n}\right\}$, $\{(-1)^n\}$, $\left\{\frac{1}{n}\right\}$, $\{n\}$, $\{-n\}$, $\{(-1)^{n+1}\}$ as examples.

17. Given the sequences $\left\{\frac{1}{n}\right\}$, $\left\{\frac{1}{n^2}\right\}$, $\left\{\frac{(-1)^n}{n}\right\}$, $\left\{\frac{1}{n+100}\right\}$, choose from these infinitesimal sequences such that (a) $\lim\limits_{n\to\infty}(x_n/y_n)=0$, (b) $\lim\limits_{n\to\infty}(x_n/y_n)=1$, (c) $\lim\limits_{n\to\infty}(x_n/y_n)=\infty$, (d) $\{x_n/y_n\}$ diverges but is bounded.

18. Given $\lim\limits_{n\to\infty} x_n = b \neq \infty$, $\lim\limits_{n\to\infty} y_n = \infty$. Prove that

(a) $\lim\limits_{n\to\infty} (x_n \pm y_n) = \infty$, (b) $\lim\limits_{n\to\infty} (x_n/y_n) = 0$,

(c) $\lim\limits_{n\to\infty} (y_n/x_n) = \infty$ $(x_n \neq 0)$, (d) $\lim\limits_{n\to\infty} (x_n y_n) = \infty$ if $b \neq 0$.

19. Prove that $\lim\limits_{n\to\infty} (x_n/y_n) = \infty$ if $\lim\limits_{n\to\infty} x_n = b \neq 0$, $\lim\limits_{n\to\infty} y_n = 0$ $(y_n \neq 0)$.

20. Test the following sequences for convergence (depending on α, β, γ):

(a) $x_n = \dfrac{n^\alpha + 1}{n^\beta + 3}$, (b) $x_n = n^\gamma \dfrac{\sqrt[3]{n^3+1} - n}{\sqrt{n+1} - \sqrt{n}}$.

21. Find the following limits:

(a) $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^2}\sin(n!)}{n+2}$, (b) $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$,

(c) $\lim\limits_{n\to\infty} \dfrac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$.

22. Let $x_n = \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{n^2+k}}$. Calculate $\lim\limits_{n\to\infty} x_n$. We evaluate x_n from above and from below:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} < \sum_{k=1}^{n} \frac{1}{\sqrt{n^2}} = 1.$$

Thus we have

$$\frac{n}{\sqrt{n^2+n}} < x_n < 1.$$

Since

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1,$$

we have $\lim\limits_{n\to\infty} x_n = 1$ in accordance with the theorem on three sequences.

On the other hand, the arbitrary term in the expression for x_n is equal to $\dfrac{1}{\sqrt{n^2+k}}$ $(k = 1, 2, \ldots, n)$.

Since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n^2+k}} = 0$, it follows that

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} \right)$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2}}$$

$$+ \ldots + \lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0 + 0 + \ldots + 0 = 0.$$

We have thus found that $1 = 0$. Find the error in the reasoning. Which of the two results is true and which is not, and why?

23. Calculate the following limits:

(a) $\lim\limits_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \ldots + \frac{n-1}{n^2} \right)$,

(b) $\lim\limits_{n\to\infty} \left(\frac{1^2}{n^3} + \frac{2^2}{n^3} + \ldots + \frac{(n-1)^2}{n^3} \right)$,

(c) $\lim\limits_{n\to\infty} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} \right)$,

(d) $\lim\limits_{n\to\infty} \left(\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} \right)$.

2.4. Remarkable Examples

I. Fundamental Concepts and Theorems

We say that the infinitely large sequence $\{x_n\}$ has a *higher order of growth* than the infinitely large sequence $\{y_n\}$ if $\{x_n/y_n\}$ is an infinitely large sequence, and in this case we shall use the designation $y_n \ll x_n$. This section is devoted to calculations of some limits which are used to compare the orders of growth of different infinitely large sequences. These limits can be found with the use of Theorem 8 on three sequences and the binomial theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \ldots + \binom{n}{k} a^{n-k}b^k + \ldots + b^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k,$$

where

$$\binom{n}{k}=\frac{n!}{k!\,(n-k)!}=\frac{n\,(n-1)\,\ldots\,(n-k+1)}{k!}\,,\quad 0!=1.$$

This formula yields an inequality

$$(a+b)^n>\frac{n\,(n-1)}{2}\,a^{n-2}b^2, \tag{1}$$

which is valid for positive a and b and for any natural number n.

The examples and problems considered in this section lead to the following results:

$$\log_{|a|} n \ll n^{\alpha} \ll a^n \ll n! \text{ for } \alpha>0,\ |a|>1. \tag{2}$$

(The relation $\log_{|a|} n \ll n^{\alpha}$ is valid for $\alpha>0$, but in Exercises 24 and 25 you have to prove it only for $\alpha \geqslant 1$.)

II. Control Questions and Assignments

1. Formulate the theorem on three sequences.
2. Write the first four terms of the binomial formula for the expansion of $(1+a)^n$.
3. Given infinitely large sequences $\{n!\}$, $\{\log_{10} n\}$, $\{\sqrt[3]{n}\}$, $\{4^n\}$, $\{n^5\}$, with the aid of relation (2) find for every two sequences which of them has a higher order of growth.

III. Worked Problems

1. Prove that $\lim\limits_{n\to\infty}\frac{n^{\alpha}}{a^n}=0$ for $\alpha>0$, $|a|>1$.

△ It follows from the definition of a limit of a sequence that if $\lim\limits_{n\to\infty}|u_n|=0$, then $\lim\limits_{n\to\infty}u_n=0$ as well. It is therefore sufficient to prove that $\lim\limits_{n\to\infty}w_n=0$, where $w_n=\frac{n^{\alpha}}{|a|^n}$. We represent w_n as

$$w_n=\left(\frac{n}{(|a|^{1/\alpha})^n}\right)^{\alpha}$$

and consider a sequence $z_n=\frac{n}{(|a|^{1/\alpha})^n}$. We shall prove that $\lim\limits_{n\to\infty}z_n=0$. Then it will follow from the result

of Example 5 in 2.1 that $\lim\limits_{n\to\infty} w_n = 0$. Since $\alpha > 0$, $|a| > 1$, it follows that $\exists \beta > 0$ such that $|a|^{1/\alpha} = 1 + \beta$. Applying inequality (1) to the binomial $(1+\beta)^n$, we get

$$(|a|^{1/\alpha})^n = (1+\beta)^n > \frac{n(n-1)}{2}\beta^2 \quad \forall n \geqslant 2.$$

It follows that

$$z_n = \frac{n}{|a|^{n/\alpha}} < \frac{2}{\beta^2(n-1)} \quad \forall n \geqslant 2.$$

We set $x_n = 0\ \forall n$ and $y_n = \frac{2}{\beta^2(n-1)}\ \forall n \geqslant 2$. The sequences $\{x_n\}$, $\{z_n\}$, $\{y_n\}$ satisfy the conditions of the theorem on three sequences since $x_n \leqslant z_n \leqslant y_n$ and $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{\alpha\to\infty} y_n = 0$. Consequently, $\lim\limits_{n\to\infty} z_n = 0$, and this is what we wished to prove.

We can thus write

$$n^\alpha \ll a^n \quad \text{for} \quad \alpha > 0, \quad |a| > 1. \ \blacktriangle$$

2. Prove that $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ for $|a| > 1$.

$\triangle$ Let us consider a sequence $z_n = \frac{|a|^n}{n!}$ and prove that $\lim\limits_{n\to\infty} z_n = 0$. Assume that k is a natural number such that $k \geqslant |a|$ and that $n > 2k$. We represent z_n as a product of n factors

$$\frac{|a|^n}{n!} = \frac{|a|}{1}\,\frac{|a|}{2}\cdots\frac{|a|}{2k}\,\frac{|a|}{2k+1}\cdots\frac{|a|}{n}.$$

Since $k \geqslant |a|$, the fraction $\frac{|a|}{2k}$ and all fractions following it are not larger than 1/2. Therefore we obtain an estimate

$$z_n = \frac{|a|^n}{n!} \leqslant |a|^{2k-1}\left(\frac{1}{2}\right)^{n-2k+1} = (2|a|)^{2k-1}\left(\frac{1}{2}\right)^n.$$

Since $z_n > 0$ and $\lim\limits_{n\to\infty} (2|a|)^{2k-1}\left(\frac{1}{2}\right)^n = 0$, according to the theorem on three sequences we have $\lim\limits_{n\to\infty} z_n = 0$. From this it follows that $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$, i.e. the se-

quence $\{n!\}$ has a higher order of growth than the sequence $\{a^n\}$ for $|a| > 1$. Thus $a^n \ll n!$ for $|a| > 1$. ▲

3. Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

△ For $n \geqslant 2$ the number $\sqrt[n]{n}$ is larger than 1. Therefore $\forall n \geqslant 2\ \exists \beta_n > 0$ such that

$$\sqrt[n]{n} = 1 + \beta_n. \tag{3}$$

It follows from relation (3) that $n = (1 + \beta_n)^n$. Applying inequality (1) to the right-hand side of the last relation, we obtain

$$n = (1+\beta_n)^n \geqslant \frac{n(n-1)}{2} \beta_n^2.$$

From this we find that $\beta_n \leqslant \sqrt{\frac{2}{n-1}}\ \forall n \geqslant 2$. According to the theorem on three sequences, it follows from the inequalities $0 < \beta_n \leqslant \sqrt{\frac{2}{n-1}}$ and the relation $\lim\limits_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$ that $\lim\limits_{n\to\infty} \beta_n = 0$. We now find from relation (3) that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. ▲

IV. Problems and Exercises for Independent Work

24. Using the definition of a limit of a sequence and the result of Example 3, prove that $\lim\limits_{n\to\infty} \frac{\log_a n}{n} = 0$ for $a > 1$.

25. Prove that $\lim\limits_{n\to\infty} \frac{\log_a n}{n^\alpha} = 0$ for $a > 1$, $\alpha > 1$.

26. Prove that the following specified sequences are infinitesimal: (a) $x_n = \frac{n}{2^n}$, (b) $x_n = \frac{3^n}{n!}$, (c) $x_n = \sqrt[n]{5} - 1$.

27. Using the result of Example 2, prove that $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$.

2.5. Monotonic Sequences

I. Fundamental Concepts and Theorems

The sequence $\{x_n\}$ is *nonincreasing* (*nondecreasing*) if $x_{n+1} \leqslant x_n$ $(x_{n+1} \geqslant x_n)$ $\forall n$.

Nonincreasing and nondecreasing sequences are known as *monotonic* sequences.

The sequence $\{x_n\}$ is *increasing* (*decreasing*) if $x_{n+1} > x_n$ $(x_{n+1} < x_n)$ $\forall n$.

Increasing and decreasing sequences are also called *strictly monotonic* sequences.

Note that a monotonic sequence is always bounded at least from one side, a nonincreasing sequence is bounded from above and a nondecreasing sequence is bounded from below, by its first term. If a monotonic sequence is also bounded from the other side, then it converges, i.e. the following theorem holds true.

Theorem 9. *A monotonic bounded sequence converges.*

II. Control Questions and Assignments

1. Formulate (a) the definition of a monotonic sequence, (b) the test for convergence of a monotonic sequence.

2. Is the boundedness of a sequence the necessary and sufficient condition of convergence of (a) a monotonic sequence, (b) an arbitrary sequence?

III. Worked Problems

Find the limit of the sequence $\{x_n\}$ which is defined by the recurrent relation:

$$x_{n+1} = x_n(2 - x_n) \quad \forall n \geqslant 1, \tag{1}$$

where x_1 is an arbitrary number satisfying the inequalities $0 < x_1 < 1$.

△ We shall first prove that the sequence $\{x_n\}$ is bounded, namely, we shall prove by induction that $\forall n$ there hold inequalities

$$0 < x_n < 1. \tag{2}$$

For x_1 inequalities (2) are satisfied by the hypothesis. Assume that inequalities (2) are valid for the number n. We shall prove that then they are also valid for the number $n + 1$. We write formula (1) in the form

$$x_{n+1} = 1 - (1 - x_n)^2.$$

It follows from inequalities (2) that $0 < (1 - x_n)^2 < 1$, and therefore $0 < x_{n+1} < 1$. We have thus proved inequalities (2) $\forall n$.

We shall now prove that the sequence $\{x_n\}$ is increasing. Since $x_n < 1$, it follows that $2 - x_n > 1$. Dividing relation (1) by x_n, we get

$$x_{n+1}/x_n = 2 - x_n > 1.$$

It follows that $x_{n+1} > x_n$ $\forall n$. Thus the sequence $\{x_n\}$ is monotonic and bounded. Consequently, according to Theorem 9, there is a limit $\lim\limits_{n\to\infty} x_n$ which will be designated as a. To find a, we pass to the limit in the recurrent formula (1). We get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} (2 - x_n), \text{ or } a = a(2 - a).$$

Hence $a = 0$ or $a = 1$. Since $x_1 > 0$ and the sequence $\{x_n\}$ is increasing, we have $a = 1$. ▲

IV. Problems and Exercises for Independent Work

28. Prove the convergence of the sequence $\{x_n\}$, where $x_n = \sum\limits_{k=1}^{n} \dfrac{1}{n+k}$.

29. Prove the convergence and calculate the limit of the sequence $\{x_n\}$ if $x_1 = \sqrt{a}$, $x_2 = \sqrt{a + \sqrt{a}}$, ..., $x_n = \sqrt{a + \sqrt{a + \ldots + \sqrt{a}}}$ (the total number of roots is n), ..., where $a > 0$.

30. Prove the convergence and calculate the limit of the sequence $\{x_n\}$ if it is defined by the recurrent relation

(a) $x_n = (x_{n-1} + x_{n-2})/2$ $\forall n \geqslant 3$, $x_1 = a$, $x_2 = b$, $a \neq b$,

b) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$ $\forall n \geqslant 1$, $a > 0$,

where x_1 is an arbitrary positive number,

(c) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$ $\forall n \geqslant 1$, $a > 0$,

where x_1 is an arbitrary negative number.

31. Prove that an unbounded monotonic sequence is infinitely large.

32. Prove the existence of the limit $\lim\limits_{n\to\infty}\left(1 + \dfrac{1}{n}\right)^n$.

2.6. Limit Points

I. Fundamental Concepts and Theorems

Let $\{x_n\}$ be a number sequence. We shall consider an arbitrary increasing sequence of positive integers k_1, $k_2, \ldots, k_n, \ldots$. Note that $k_n \geqslant n$. From $\{x_n\}$ we choose terms with the numbers $k_1, k_2, \ldots, k_n, \ldots$:

$$x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots .$$

The number sequence $\{x_{k_n}\}$ obtained is a *subsequence* of the sequence $\{x_n\}$.

Theorem 10. *If* $\lim\limits_{n\to\infty} x_n = a$, *then any sequence* $\{x_{k_n}\}$ *converges to* a *as* $n \to \infty$.

Theorem 11 (Bolzano-Weierstrass theorem). *Any bounded sequence always contains a convergent subsequence.*

Definition 1. *The number* a *is a* ***limit point*** *of the sequence* $\{x_n\}$ *if the sequence* $\{x_n\}$ *contains a subsequence* $\{x_{k_n}\}$ *converging to* a.

We can give another, equivalent, definition of a limit point.

Definition 2. *The number* a *is a* ***limit point*** *of the sequence* $\{x_n\}$ *if in any* ε*-neighbourhood of the point* a *there are infinitely many terms of the sequence* $\{x_n\}$.

Remark 1. It follows from Theorem 10 that a convergent sequence has only one limit point which coincides with its limit.

Remark 2. It follows from Theorem 11 that every bounded sequence has at least one limit point.

The largest (smallest) limit point of the sequence $\{x_n\}$ bounded from above (from below), is the *limit superior* (*limit inferior*) of the sequence and is designated as $\overline{\lim\limits_{n\to\infty}}\, x_n$ $\left(\underline{\lim\limits_{n\to\infty}}\, x_n\right)$.

It is evident that if $\{x_n\}$ is convergent, then $\overline{\lim\limits_{n\to\infty}}\, x_n = \underline{\lim\limits_{n\to\infty}}\, x_n = \lim\limits_{n\to\infty} x_n$. If the sequence $\{x_n\}$ is not bounded from above (from below), then we set $\overline{\lim\limits_{n\to\infty}}\, x_n = +\infty$ $\left(\underline{\lim\limits_{n\to\infty}}\, x_n = -\infty\right)$.

II. Control Questions and Assignments

1. Formulate the definition of (a) a sequence, (b) a limit point (give two definitions and prove their equivalence), (c) the upper (lower) limit of a sequence.

2. Give a geometric interpretation of the definition of a limit point.

3. Is the limit of a sequence its limiting point? Substantiate the answer.

4. Given the sequences $\{n(-1)^n + 1\}$, $\{n\}$, $\{(-1)^n + 1\}$, indicate the sequence which (a) has a limit point, (b) does not have a limit point, (c) has two limit points, (d) has only one limit point. Are there convergent sequences among them?

5. Prove that a convergent sequence has only one limit point and that point coincides with its limit. Find whether the converse statement is true: "If a sequence has a single limit point, then it is convergent".

6. Given a sequence $\{x_n\}$ and it is known that any neighbourhood of the point a contains infinitely many terms of the sequence and none of the intervals to which the point a does not belong, does not contain infinitely many terms of the sequence. Does it follow that $\lim\limits_{n\to\infty} x_n = a$?

7. Let $\lim\limits_{n\to\infty} x_n^2 = 4$. Can the sequence $\{x_n\}$ be (a) convergent (if it can, then what is its limit?), (b) divergent?

8. Formulate the Bolzano-Weierstrass theorem.

9. Find whether the following statement is true: "If a sequence is unbounded, then a convergent sequence can be isolated from it".

III. Worked Problems

1. Prove the divergence of the sequence $x_n = (-1)^n$.

△ Consider two subsequences of this sequence: $x_{2k} = 1$ and $x_{2k-1} = -1$ ($k = 1, 2, \ldots$). It is evident that $\lim\limits_{k\to\infty} x_{2k} = 1$, $\lim\limits_{k\to\infty} x_{2k-1} = -1$.

Thus the sequence $\{(-1)^n\}$ has two limit points, 1 and -1, and therefore cannot be convergent since a convergent sequence has only one limit point. ▲

2. Find (a) all limit points of the sequence $\{\sin n^\circ\}$, (b) the upper and the lower limit of this sequence.

△ (a) Each of the numbers 0, $\pm\sin 1°$, $\pm\sin 2°$, . . ., $\pm\sin 89°$, ± 1 appears in the sequence infinitely many times since $\sin n° = \sin(360°p + n°)$ $\forall n$, $p \in \mathbf{N}$. Therefore each of the indicated numbers is a limit point of the sequence $\{\sin n°\}$. The sequence has no other limit points since if the number a does not coincide with any one of these 181 numbers, then there is a neighbourhood of the point a which does not contain any terms of the sequence.

(b) From the 181 limit points indicated in (a) the point -1 is the least and the point 1 is the greatest, i.e. $\overline{\lim\limits_{n\to\infty}} \sin n° = 1$, $\underline{\lim\limits_{n\to\infty}} \sin n° = -1$. ▲

3. Find (a) all limit points of the sequence 0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, . . ., (b) the upper and the lower limit of this sequence.

△ (a) The sequence $\{x_n\}$ is the set of all rational numbers belonging to the interval [0, 1] which appear in the indicated order. Since any ε-neighbourhood of any real number of the interval [0, 1] contains infinitely many rational numbers (see Exercise 5 in 1.1), every point of this interval is a limit point of the given sequence. Now if the point $a \notin [0, 1]$, then it is not a limit point of the given sequence since there is a neighbourhood of such a point, which does not contain any term of the sequence.

(b) It is evident that $\underline{\lim\limits_{n\to\infty}}\, x_n = 0$, $\overline{\lim\limits_{n\to\infty}}\, x_n = 1$. ▲

4. Prove that the infinitely large sequence $\{x_n\}$ does not have a limit point.

△ We use an indirect proof. Assume that a point a is a limit point of the sequence $\{x_n\}$. Then there is a subsequence $\{x_{k_n}\}$ such that $\lim\limits_{n\to\infty} x_{k_n} = a$. On the other hand, $\{x_{k_n}\}$ has no limit since it is infinitely large. Indeed, since $\{x_n\}$ is infinitely large, it follows that $\forall A > 0$ $\exists N$: $\forall n > N$ $|x_n| > A$. Since $k_n \geqslant n$ and $k_{n+1} > k_n$, it follows that $\forall k_n > N$: $|x_{k_n}| > A$, i.e. $\{x_{k_n}\}$ is infinitely large. The contradiction obtained proves that $\{x_n\}$ does not have a limit point. ▲

IV. Problems and Exercises for Independent Work

33. Prove that

(a) an infinitely large subsequence can be isolated from any unbounded sequence,

(b) any subsequence of an infinitely large sequence is infinitely large,

(c) a monotonic unbounded sequence does not have a limit point,

(d) every bounded sequence has an upper and a lower limit.

34. Given that the sequences $\{x_n\}$ and $\{y_n\}$ have a limit point each, give examples showing that the sequences $\{x_n + y_n\}$ and $\{x_n \cdot y_n\}$ may (a) have no limit points, (b) have one limit point each, (c) have two limit points each.

35. Find all limit points of the sequence $\{x_n\}$, also find $\overline{\lim\limits_{n\to\infty}} x_n$ and $\underline{\lim\limits_{n\to\infty}} x_n$ if

(a) $x_n = (-1)^{n-1}\left(2+\frac{3}{n}\right)$, (b) $x_n = 1+\frac{n}{n+1}\cos\frac{\pi n}{2}$,

(c) $x_n = 1+2(-1)^{n+1}+(-1)^{n(n-1)/2}$,

(d) $x_n = \frac{n-1}{n+1}\cos\frac{2\pi n}{3}$,

(e) $x_n = 1+n\sin\frac{\pi n}{2}$, (f) $x_n = \frac{(-1)^n}{n}+\frac{1+(-1)^n}{2}$,

(g) $x_n = \left(1+\frac{1}{n}\right)^n\cdot(-1)^n+\sin\frac{\pi n}{4}$,

(h) $x_n = \frac{n}{n+1}\sin^2\frac{\pi n}{4}$, (i) $x_n = \sqrt[n]{1+2^{n(-1)^n}}$,

(j) $x_n = \cos^n\frac{2\pi n}{3}$, (k) $x_n = (-1)^n n$.

36. Test the sequence

$$x_n = \frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\ldots+(-1)^{n-1}\frac{n}{n}$$

for convergence.

2.7. Fundamental Sequences. Cauchy Condition for Convergence of a Sequence

I. Fundamental Concepts and Theorems

Definition 1. *The sequence $\{x_n\}$ is* ***fundamental*** *if* $\forall\varepsilon > 0$ $\exists N$ *such that* $\forall n > N$ and $\forall$ *natural number p the inequality* $|x_n - x_{n+p}| < \varepsilon$ *holds true.*

This definition is equivalent to the following one.

Definition 2. *The sequence $\{x_n\}$ is* ***fundamental*** *if*

$\forall \varepsilon > 0\ \exists N$ *such that* $\forall n > N$ and $\forall m > N$ *the inequality* $|x_n - x_m| < \varepsilon$ *holds true.*

The geometric interpretation of these definitions is as follows: if the sequence $\{x_n\}$ is fundamental, then $\forall \varepsilon > 0\ \exists N$ such that the distance between any two terms of the sequence with numbers larger than N is smaller than ε.

Theorem 12 (Cauchy condition for convergence of a sequence). *For a sequence to be convergent, it is necessary and sufficient for it to be fundamental.*

II. Control Questions and Assignments

1. Formulate the definition of

(a) a fundamental sequence (give two definitions and prove their equivalence),

(b) a nonfundamental sequence (using the rule of constructing negations).

Give a geometric interpretation of these definitions.

2. Formulate Cauchy condition for convergence of a sequence.

III. Worked Problems

1. Using Cauchy condition, prove the convergence of the sequence $\{x_n\}$, where $x_n = \sum_{k=1}^{n} \frac{\sin k}{k^2}$.

△ By virtue of Cauchy condition, it is sufficient to prove that the sequence $\{x_n\}$ is fundamental. For that purpose we evaluate $|x_n - x_{n+p}|$. We have

$$|x_n - x_{n+p}| = \left| \sum_{k=n+1}^{n+p} \frac{\sin k}{k^2} \right| \leqslant \sum_{k=n+1}^{n+p} \frac{1}{k^2}.$$

Since $\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$, it follows that

$$\sum_{k=n+1}^{n+p} \frac{1}{k^2} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+p)^2}$$

$$< \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$+ \ldots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right) = \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.$$

Therefore $\forall n, p \in N$, we have

$$|x_n - x_{n+p}| < 1/n. \tag{1}$$

We specify an arbitrary $\varepsilon > 0$ and set $N = [1/\varepsilon]$. Then $\forall n > N$ the inequality $n \geqslant [1/\varepsilon] + 1 > 1/\varepsilon$ holds true, whence $1/n < \varepsilon$. Consequently, using inequality (1), $\forall n > N$ and $\forall$ natural p, we get $|x_n - x_{n+p}| < 1/n < \varepsilon$. This proves that the sequence is fundamental. ▲

2. Using Cauchy condition, prove the divergence of the sequence $\{x_n\}$, where $x_n = \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k}}$.

△ By virtue of Cauchy condition, it is sufficient to prove that the sequence $\{x_n\}$ is not fundamental. For that purpose we evaluate $|x_n - x_{n+p}|$. We have

$$|x_n - x_{n+p}| = \sum_{k=n+1}^{n+p} \frac{1}{\sqrt{k}} \geqslant \frac{p}{\sqrt{n+p}} \quad \forall n,\ p \in \mathbf{N}.$$

In particular, for $p = n$ we get

$$|x_n - x_{2n}| \geqslant \frac{\sqrt{n}}{\sqrt{2}} \geqslant \frac{1}{\sqrt{2}} \quad \forall n. \tag{2}$$

We take $\varepsilon = 1/\sqrt{2}$. Then $\forall N\ \exists\ n > N$ and a natural p such that $|x_n - x_{n+p}| \geqslant \varepsilon$. Indeed, by virtue of inequality (2), it is sufficient to take any $n > N$ and $p = n$. This proves that the sequence $\{x_n\}$ is not fundamental. ▲

IV. Problems and Exercises for Independent Work

37. Using Cauchy's condition, prove the convergence of the sequence $\{x_n\}$ if

(a) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$, (b) $x_n = \sum\limits_{k=1}^{n} \frac{\cos(k!)}{k(k+1)}$, (c) $x_n = \sum\limits_{k=1}^{n} \frac{\sin k}{2^k}$,

(d) $x_n = \sum\limits_{k=0}^{n} a_k q^k$, where $|q| < 1$ and $|a_k| \leqslant M\ \forall k$, $M > 0$.

38. Using Cauchy condition, prove that if the sequence $x_n = \sum\limits_{k=1}^{n} a_k$ is convergent, then $\lim\limits_{n\to\infty} a_n = 0$.

39. Using Cauchy condition, prove the divergence of the sequence $\{x_n\}$ if

(a) $x_n = \sum\limits_{k=1}^{n} (-1)^k$, (b) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$.

Chapter 3

Limit of a Function. Continuity of a Function

3.1. Limit of a Function. Limit Theorems. Infinitely Large Functions

I. Fundamental Concepts and Theorems

1. Limit of a function at a point. Assume that x is a numerical variable and X is its range. If we put a number y into correspondence with every number $x \in X$ then we say that a *function* is defined on the set X and write $y = f(x)$. The variable x is an *independent variable* (or the *argument of the function*), the set X is the *domain of definition* (or simply *domain*) *of the function* $f(x)$, and the number y which corresponds to this argument x is a *particular value of the function at the point* x. The set Y of all particular values of the function is the *set of values* (*range*) *of the function* $f(x)$.

The point a ($a \in X$ or $a \notin X$) is a *limit point* of the set X if in any neighbourhood of the point a there are points of the set X different from a.

In the definitions given in this section we assume that a is a limit point of the set X which is the domain of definition of the function $f(x)$.

Definition 1 (Cauchy's). *The number b is the limit of the function $f(x)$* ***at the point*** *a (as $x \to a$) if $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x$, satisfying the conditions $x \in X$, $0 < |x - a| < \delta$, the inequality $|f(x) - b| < \varepsilon$ holds true.*

Definition 2 (Heine's). *The number b is* ***the limit of the function*** *$f(x)$* ***at the point*** *a if for any sequence $\{x_n\}$ converging to a and such that $x_n \in X$, $x_n \neq a$, the corresponding sequence of values of the function $\{f(x_n)\}$ converges to b.*

The notation is $\lim\limits_{x \to a} f(x) = b$, or $f(x) \to b$ as $x \to a$.

It should be emphasized that the concept of the limit of a function of a point a is introduced only for the limit points a of the domain of the function. Note that in that case the function may not be defined at the point a, i.e., in general, $a \notin X$.

Here are the formulations of the negations of Definitions 1 and 2.

Negation of Definition 1. The number b is not a limit of the function $f(x)$ at the point a $[b \neq \lim\limits_{x\to a} f(x)]$ if $\exists \varepsilon > 0$ such that $\forall \delta > 0$ $\exists x \in X$ for which $0 < |x - a| < \delta$ and $|f(x) - b| \geqslant \varepsilon$.

Negation of Definition 2. The number $b \neq \lim\limits_{x\to a} f(x)$ if there is a sequence $\{x_n\}$ $(x_n \in X,\ x_n \neq a)$ which converges to a and is such that the corresponding sequence $\{f(x_n)\}$ does not converge to b.

2. Limit theorems.

Theorem 1. *Definitions* 1 *and* 2 *of the limit of a function are equivalent.*

Theorem 2. *Assume that* $f(x)$ *and* $g(x)$ *are defined in a neighbourhood of the point* a *except, maybe, for the point* a *itself, and* $\lim\limits_{x\to a} f(x) = b$, $\lim\limits_{x\to a} g(x) = c$. *Then*

$$\lim_{x\to a} (f(x) + g(x)) = b + c,$$

$$\lim_{x\to a} (f(x) - g(x)) = b - c,$$

$$\lim_{x\to a} f(x)\, g(x) = bc,$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c} \quad \textit{provided that } c \neq 0.$$

Theorem 3. *Assume that the functions* $f(x)$, $g(x)$ *and* $h(x)$ *are defined in a neighbourhood of the point* a, *except, maybe, for the point* a *itself, and satisfy the inequalities* $f(x) \leqslant g(x) \leqslant h(x)$. *Let* $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = b$. *Then* $\lim\limits_{x\to a} g(x) = b$.

3. One-sided limits.

Definition 1 (Cauchy's). *The number* b *is the* ***right-hand (left-hand) limit*** *of the function* $f(x)$ *at the point* a *if* $\forall \varepsilon > 0$ $\exists \delta > 0$ *such that* $\forall x$, *which satisfies the conditions* $x \in X$, $a < x < a + \delta$ $(a - \delta < x < a)$, *the inequality* $|f(x) - b| < \varepsilon$ *holds true.*

Definition 2 (Heine's). *The number* b *is the* ***right-hand (left-hand) limit*** *of the function* $f(x)$ *at the point* a *if for any sequence* $\{x_n\}$ *which converges to* a *and is such that* $x_n \in X$, $x_n > a$ $(x_n < a)$, *the corresponding sequence of the values of the function* $\{f(x_n)\}$ *converges to* b.

The notations are $\lim\limits_{x\to a+0} f(x) = b$ or $f(a + 0) = b$

($\lim\limits_{x \to a-0} f(x) = b$ or $f(a-0) = b$ respectively).

Definitions 1 and 2 are equivalent.

Theorem 4. *If there are $f(a+0)$ and $f(a-0)$ and $f(a+0) = f(a-0) = b$, then there is a limit* $\lim\limits_{x \to a} f(x) = b$.

Theorem 5. *If the function $f(x)$ is defined in a neighbourhood of the point a, except, maybe, for the point a itself, and there is a limit $\lim\limits_{x \to a} f(x) = b$, then there are $f(a+0)$ and $f(a-0)$ with $f(a+0) = f(a-0) = b$.*

4. Limit of a function as $x \to \infty$. Let the function $f(x)$ be defined on the ray $(c, +\infty)$.

Definition 1 (Cauchy's). *The number b is the* ***limit of the function*** *$f(x)$ as $x \to +\infty$ $[b = \lim\limits_{x \to +\infty} f(x)]$ if $\forall \varepsilon > 0$ $\exists A > 0$ $(A \geqslant c)$ such that $\forall x > A$ there holds an inequality $|f(x) - b| < \varepsilon$.*

Definition 2 (Heine's). *The number $b = \lim\limits_{x \to +\infty} f(x)$ if for any infinitely large sequence $\{x_n\}$ $(x_n > c)$ the corresponding sequence of the values of the function $\{f(x_n)\}$ converges to b.*

Definitions 1 and 2 are equivalent.

By analogy we can define $\lim\limits_{x \to -\infty} f(x)$. If $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to +\infty} f(x) = b$, then we write $\lim\limits_{x \to \infty} f(x) = b$. For instance, $\lim\limits_{x \to \infty} (1/x) = 0$.

A theorem analogous to Theorem 2 is valid for one-sided limits and the limits as $x \to \infty$.

5. Infinitely large functions.

Definition 1. *The function $f(x)$ is* ***infinitely large on the right of the point*** *a if $\forall M > 0$ $\exists \delta > 0$ such that $\forall x$ satisfying the condition $x \in X$, $a < x < a + \delta$, there holds an inequality*

$$|f(x)| > M. \tag{1}$$

The notation is $\lim\limits_{x \to a+0} f(x) = \infty$, or $f(a+0) = \infty$. It should be emphasized that the notation only means that $f(x)$ is infinitely large on the right of the point a but does not mean at all that $f(x)$ has a right-hand limit at the point a. This limit evidently does not exist.

If the inequality $f(x) > M$ ($f(x) < -M$) is satisfied in Definition 1 instead of inequality (1), then we say that the function $f(x)$ is infinitely large of the plus (minus) sign on the right of the point a and write $\lim\limits_{x\to a+0} f(x) = +\infty$, or $f(a+0) = +\infty$ ($\lim\limits_{x\to a+0} f(x) = -\infty$, or $f(a+0) = -\infty$ respectively).

An infinitely large function on the left of the point a is defined by analogy.

If a function is infinitely large on the right and on the left of the point a, then we write $\lim\limits_{x\to a} f(x) = \infty$. For instance, $\lim\limits_{x\to 0} (1/x) = \infty$.

Definition 2. *The function $f(x)$ is* ***infinitely large on the right (on the left) of the point a*** *if for any sequence $\{x_n\}$ converging to a and such that $x_n \in X$, $x_n > a$ $(x_n < a)$, the corresponding sequence $\{f(x_n)\}$ is infinitely large.*

Definitions 1 and 2 are equivalent.

Let the function $f(x)$ be defined on the ray $(c, +\infty)$.

Definition 3. *The function $f(x)$ is* ***infinitely large as*** *$x\to+\infty$ if $\forall M > 0$ $\exists A$ $(A \geqslant c)$ such that $\forall x > A$: $|f(x)| > M$.*

Definition 4. *The function $f(x)$ is* ***infinitely large as*** *$x\to+\infty$ if for any infinitely large sequence $\{x_n\}$ $(x_n > c)$ the corresponding sequence $\{f(x_n)\}$ is infinitely large.*

The notation is $\lim\limits_{x\to+\infty} f(x) = \infty$.

Definitions 3 and 4 are equivalent.

By analogy, we can introduce the concept of an infinitely large function as $x\to-\infty$: $\lim\limits_{x\to-\infty} f(x) = \infty$. If the function $f(x)$ is infinitely large as $x\to+\infty$ and as $x\to-\infty$, then we write $\lim\limits_{x\to\infty} f(x) = \infty$. For instance, $\lim\limits_{x\to\infty} x = \infty$.

II. Control Questions and Assignments

1. Formulate two definitions of the limit of a function at a point. What does the equivalence of these definitions mean?

2. Using Heine's definition of the limit of a function, prove the uniqueness of the limit of a function at a point.

3. Prove that $\forall x_0$ $\lim\limits_{x\to x_0} x = x_0$ using (a) Cauchy's, (b) Heine's, definition of the limit of a function.

4. Given a function $f(x) = \frac{|x|}{x}$, find whether the function $f(x)$ is defined at the point $x = 0$. Is the point $x = 0$ a limit point of the domain of the function? Is there a limit $\lim\limits_{x\to 0} f(x)$?

5. Formulate the negations of two definitions of the limit of a function at a point.

6. Formulate Theorems 2 and 3 on the limits of functions.

7. Formulate two definitions of one-sided limits of a function and the negations of these definitions.

8. Are there $f(3+0)$ and $f(3-0)$ if $f(x) = \frac{|3-x|}{3-x}$? Is there $\lim\limits_{x\to 3} f(x)$?

9. Under what conditions does the existence of one-sided limits (a limit of a function) imply the existence of a limit of a function (one-sided limits)?

10. Formulate two definitions of a limit of a function as $x \to +\infty$ and the negations of these definitions.

11. Prove that the function $f(x) = x$ has no limit as $x \to +\infty$.

12. Prove that $\lim\limits_{x\to+\infty} x = +\infty$.

13. Formulate the Cauchy and Heine definitions corresponding to the following symbolic notations:

(a) $\lim\limits_{x\to a} f(x) = \infty$, (b) $f(a+0) = -\infty$, (c) $\lim\limits_{x\to-\infty} f(x) = b$,

(d) $\lim\limits_{x\to\infty} f(x) = \infty$, (e) $\lim\limits_{x\to-\infty} f(x) = +\infty$.

14. Prove that the function $f(x) = 1/(x-3)$ is infinitely large at the point $x = 3$.

III. Worked Problems

1. Prove that $\lim\limits_{x\to 0} \sin x = 0$.

△ We use the inequality $|\sin x| \leqslant |x|$ $\forall x$. We specify an arbitrary $\varepsilon > 0$ and set $\delta = \varepsilon$. Then, if $|x| < \delta$, then $|\sin x| \leqslant |x| < \delta = \varepsilon$. And this means (according to Cauchy's definition of a limit of a function) that $\lim\limits_{x\to 0} \sin x = 0$. ▲

2. Calculate $\lim\limits_{x\to 1} f(x)$, where $f(x) = \frac{x^2-1}{x-1}$.

△ Since $\lim\limits_{x\to 1}(x^2-1) = 0$ and $\lim\limits_{x\to 1}(x-1) = 0$, it follows

that this limit is an indeterminate form of type 0/0 and we cannot use Theorem 2 on the limit of the quotient of two functions. We use the fact that when the limit of the function at the point $x = 1$ is considered, its argument does not assume a value 1. Therefore, $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(x+1)$ since $f(x) = \frac{x^2-1}{x-1} = x+1$ if $x \neq 1$. Let $\{x_n\}$ be an arbitrary sequence converging to 1 $(x_n \neq 1)$ and then $\lim\limits_{n\to\infty}(x_n+1) = 2$. This means (according to Heine's definition of the limit of a function) that $\lim\limits_{x\to 1}(x+1) = \lim\limits_{x\to 1} f(x) = 2$. ▲

3. Calculate the limit $\lim\limits_{x\to 1} \frac{\sqrt{3+x}-2}{x-1}$.

△ As in Example 2, this limit is an indeterminate form of type 0/0. However, as distinct from Example 2, here we cannot directly "cancel" $x - 1$ out of the numerator and the denominator. Therefore we first transform the function by multiplying the numerator and denominator by $(\sqrt{3+x}+2)$, i.e. by an expression conjugate to the numerator. We obtain

$$\frac{\sqrt{3+x}-2}{x-1} = \frac{x-1}{(x-1)(\sqrt{3+x}+2)}.$$

Since, when this limit is considered, the argument x does not assume the value $x = 1$, we get the following expression when cancelling out $x - 1$:

$$\lim_{x\to 1}\frac{\sqrt{3+x}-2}{x-1} = \lim_{x\to 1}\frac{1}{\sqrt{3+x}+2} = \frac{1}{4}. \ ▲$$

4. Prove that the Dirichlet function

$$D(x) = \begin{cases} 0 \text{ if } x \text{ is an irrational number,} \\ 1 \text{ if } x \text{ is a rational number} \end{cases}$$

does not have a limit at any point.

△ We shall prove that at an arbitrary point a the function $D(x)$ does not satisfy Heine's definition of a limit of a function. To do that, we indicate two sequences $\{x_n\}$ and $\{x'_n\}$ which converge to a and are such that $\lim\limits_{n\to\infty} D(x_n) \neq \lim\limits_{n\to\infty} D(x'_n)$. We shall first consider the sequence $\{x_n\}$ of rational points, which converges to a.

For this sequence $D(x_n) = 1 \ \forall n$ and therefore $\lim\limits_{n\to\infty} D(x_n) = 1$. Then we consider the sequence $\{x'_n\}$ of irrational points, which converges to a. For this sequence $D(x'_n) = 0 \ \forall n$ and therefore $\lim\limits_{n\to\infty} D(x'_n) = 0$. Thus $\lim\limits_{n\to\infty} D(x_n) \neq \lim\limits_{n\to\infty} D(x'_n)$. Hence it follows that the limit of the function $D(x)$ at the point a does not exist. ▲

5. Let us consider the set of all irrational numbers belonging to the interval $(-1, 1)$. We designate it as I_{ir}. We define the function $f(x)$ on the set I_{ir}: $f(x) = 1$ if $x \in I_{ir}$. Prove that $\lim\limits_{x\to a} f(x) = 1$, where a is an arbitrary point from the interval $[-1, 1]$ (rational or irrational).

△ Let $a \in [-1, 1]$. The point a is a limit point of the set I_{ir}. We use Heine's definition of a limit of a function. Assume that $\{x_n\}$ is an arbitrary sequence of points of the set I_{ir}, which converges to the point a $(x_n \neq a)$. By the hypothesis $f(x_n) = 1 \ \forall x_n \in I_{ir}$, and therefore $\lim\limits_{n\to\infty} f(x_n) = 1$, and, consequently, $\lim\limits_{x\to a} f(x) = 1$. ▲

6. Prove that the function $\sin x$ does not have a limit as $x \to +\infty$:

△ We shall prove that this function does not satisfy Heine's definition of the limit of a function as $x \to +\infty$. To do that, we indicate an infinitely large sequence $\{x_n\}$ such that the sequence $\{\sin x_n\}$ diverges. We set $x_n = \frac{\pi}{2}(2n + 1)$. Then $\lim\limits_{n\to\infty} x_n = +\infty$ and the sequence $\{\sin x_n\} = -1, 1, -1, 1, \ldots$ diverges. It follows that the function $\sin x$ does not have a limit as $x \to +\infty$. ▲

7. Let

$$f(x) = \begin{cases} x & \text{for } x < 0, \\ \sin x & \text{for } x > 0 \end{cases}$$

($f(x)$ is not defined for $x = 0$). Is there a limit $\lim\limits_{x\to 0} f(x)$?

△ We calculate the one-sided limits of the function $f(x)$ at the point $x = 0$ using Theorem 5 for the functions $y = \sin x$ and $y = x$ at the point $a = 0$:

$$f(a+0) = \lim_{x\to 0+0} \sin x = \lim_{x\to 0} \sin x = 0,$$
$$f(a-0) = \lim_{x\to 0-0} x = \lim_{x\to 0} x = 0.$$

According to Theorem 4, it follows that there is a limit $\lim\limits_{x\to 0} f(x)$ and it is equal to zero. ▲

8. Calculate $\lim\limits_{x\to\infty} f(x)$, where $f(x)=\frac{100x^2+1}{x^2+100}$.

△ This limit is an indeterminate form ∞/∞ since the numerator and the denominator are infinitely large functions as $x\to\infty$. We represent $f(x)$ in the form

$$f(x)=\frac{100+1/x^2}{1+100/x^2}\quad (x\neq 0).$$

Since $\lim\limits_{x\to\infty}(1/x^2)=0$, we can use Theorem 2 (as $x\to\infty$) and obtain

$$\lim_{x\to\infty} f(x)=\frac{\lim\limits_{x\to\infty}(100+1/x^2)}{\lim\limits_{x\to\infty}(1+100/x^2)}=\frac{100}{1}=100. \ ▲$$

Examples 2 and 8 allow us to formulate general rules for calculating limits of the form $\lim\limits_{x\to a} R(x)$ and $\lim\limits_{x\to\infty} R(x)$. Here $R(x)$ is a rational function (rational fraction), i.e. $R(x)=P_n(x)/Q_m(x)$, where $P_n(x)$ and $Q_m(x)$ are polynomials of degrees n and m respectively.

If $\lim\limits_{x\to a} Q_m(x)=Q_m(a)\neq 0$, then $\lim\limits_{x\to a} R(x)=\frac{P_n(a)}{Q_m(a)}$.

If $\lim\limits_{x\to a} Q_m(x)=0$ and $\lim\limits_{x\to a} P_n(x)\neq 0$, then $\lim\limits_{x\to a} R(x)=\infty$.

If $\lim\limits_{x\to a} Q_m(x)=\lim\limits_{x\to a} P_n(x)$, then $P_n(x)=(x-a)\,P^*_{n-1}(x)$, $Q_m(x)=(x-a)\,Q^*_{m-1}(x)$ and

$$\lim_{x\to a} R(x)=\lim_{x\to a}\frac{P^*_{n-1}(x)}{Q^*_{m-1}(x)}.$$

To calculate $\lim\limits_{x\to\infty} R(x)$, we must divide the numerator and the denominator of the function $R(x)$ by x^m and then calculate the limit of the resulting function, taking into account that $\lim\limits_{x\to\infty}\frac{Q_m(x)}{x^m}=b_0$, where b_0 is a coefficient of x^m of the polynomial $Q_m(x)$.

9. Let $\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{x\to a} g(x)=+\infty$. Prove that $\lim\limits_{x\to a}(f(x)+g(x))=+\infty$.

$\triangle$ We shall prove that the function $f(x) + g(x)$ satisfies the definition of an infinitely large function of the plus sign at the point a, i.e. $\forall M > 0 \ \exists \delta > 0$ such that $\forall x$ satisfying the condition $0 < |x - a| < \delta$ the inequality $f(x) + g(x) > M$ holds true. Since $\lim\limits_{x \to a} f(x) = b$, there is a δ_1-neighbourhood of the point a in which (for $x \neq a$)

$$|f(x)| < C, \tag{2}$$

where C is a positive number (prove this yourself!). We specify an arbitrary $M > 0$. Since $\lim\limits_{x \to a} g(x) = +\infty$, it follows that for the number $M + C \ \exists \delta > 0 \quad (\delta \leqslant \delta_1)$ such that $\forall x$ satisfying the condition $0 < |x - a| < \delta$ there holds an inequality

$$g(x) > M + C. \tag{3}$$

From inequalities (2) and (3) we find that for $0 < |x - a| < \delta \leqslant \delta_1$ there holds an inequality

$$f(x) + g(x) \geqslant g(x) - |f(x)| > M + C - C = M,$$

and this is what we wished to prove.

■

IV. Problems and Exercises for Independent Work

1. Using Cauchy's definition of a limit of a function, prove that $\lim\limits_{x \to 0} (x \sin (1/x)) = 0$.

2. Prove that the function $f(x) = \sin (1/x)$ does not have a limit at the point $x = 0$.

3. Is there a limit $\lim\limits_{x \to 0} \{x\}$, where $\{x\} = x - [x]$ is the fractional part of the number x?

4. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is an irrational number,} \\ 1 & \text{if } x \text{ is a rational number.} \end{cases}$

Prove that $f(x)$ has a limit at the points $x = 1$ and $x = -1$ and does not have a limit at the other points.

5. Prove that (a) $\lim\limits_{x \to 0} (1 - \cos x) = 0$, (b) $\lim\limits_{x \to 0} \tan x = 0$.

6. (a) Using the inequality $\sin x < x < \tan x \quad (0 < x < \pi/2)$ and Theorem 3, prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \textbf{(the first remarkable limit).}$$

(b) Using the first remarkable limit, prove that

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}, \qquad \lim_{x\to 0}\frac{\tan x}{x}=1.$$

7. Assume that

$$R(x)=\frac{a_0x^n+a_1x^{n-1}+\ldots+a^n}{b_0x^m+b_1x^{m-1}+\ldots+b_m}, \qquad a_0\neq 0,\ b_0\neq 0.$$

Prove that

$$\lim_{x\to\infty}R(x)=\begin{cases}\infty & \text{for } n>m,\\ a_0/b_0 & \text{for } n=m,\\ 0 & \text{for } n<m.\end{cases}$$

8. Calculate the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{(1+x)^3-(1+3x+3x^2)}{x^4+x^3}$, (b) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^3-2x^2+x-2}$,

(c) $\lim\limits_{x\to 3}\dfrac{x^2-5x+6}{x^2-8x+15}$, (d) $\lim\limits_{x\to 1}\dfrac{x^4-3x+2}{x^5-4x+3}$,

(e) $\lim\limits_{x\to 1}\dfrac{x^m-1}{x-1}$ (m is a natural number).

9. Calculate the following limits:

(a) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}$, (b) $\lim\limits_{x\to -8}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$,

(c) $\lim\limits_{x\to 16}\dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$.

10. Calculate the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{x^2-4}{(x-2)(x+1)}$, (b) $\lim\limits_{x\to\infty}\dfrac{(x-3)^{10}(5x+1)^{10}}{(3x^2-2)^{25}}$,

(c) $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}}$, (d) $\lim\limits_{x\to+\infty}\dfrac{\sqrt[5]{x}+\sqrt[4]{x}+\sqrt[3]{x}}{\sqrt[3]{2x+1}}$.

11. Prove that $\lim\limits_{x\to\infty}\cos x$ does not exist.

12. Is there a limit $\lim\limits_{x\to a}f(x)$ if

(a) $a=1$, $f(x)=x\,\mathrm{sgn}\,(x-1)$, where

$$\mathrm{sgn}\,x=\begin{cases}1 & \text{for } x>0,\\ 0 & \text{for } x=0,\\ -1 & \text{for } x<0,\end{cases}$$

(b) $a=0, \quad f(x)=\begin{cases} \dfrac{1-\cos x}{x^2} & \text{for } x<0, \\ \dfrac{x}{2x+x^2} & \text{for } x>0, \end{cases}$

(c) $a=0, \quad f(x)=\begin{cases} \dfrac{\sin x}{x} & \text{for } x<0, \\ \cos x & \text{for } x\geqslant 0? \end{cases}$

13. Let $\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{x\to a} g(x)=+\infty$. Prove that

(a) $\lim\limits_{x\to a} (f(x)-g(x))=-\infty$,

(b) $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)}=\infty$ (for $b\neq 0$),

(c) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}=0$, (d) $\lim\limits_{x\to a} f(x)\,g(x)=\infty$ (for $b\neq 0$).

14. Let $\lim\limits_{x\to a} f(x)=0$ (with $f(x)\neq 0$ for $x\neq a$), $\lim\limits_{x\to a} g(x)=b\neq 0$. Prove that $\lim\limits_{x\to a} \dfrac{g(x)}{f(x)}=\infty$.

3.2. Continuity of a Function at a Point

I. Fundamental Concepts and Theorems

1. **Continuity of a function at a point.** Let a function be defined in a neighbourhood of the point a.

Definition. *The function $f(x)$ is* ***continuous at the point*** a *if* $\lim\limits_{x\to a} f(x) = f(a)$.

Assume that the function $f(x)$ is defined in the right-hand (left-hand) half-neighbourhood of the point a, i.e. on a half-open interval $[a, a+\varepsilon)$ ($(a-\varepsilon, a]$ respectively). The function $f(x)$ is *continuous on the right* (*on the left*) of the point a if $f(a+0) = f(a)$ ($f(a-0) = f(a)$ respectively).

Theorem 6. *For a function to be continuous at the point a, it is necessary and sufficient for it to be continuous on the right and left of that point.*

Theorem 7. *If the functions $f(x)$ and $g(x)$ are continuous at the point a, then the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x)\,g(x)$ and $f(x)/g(x)$ are also continuous at the point a (the quotient is continuous provided that $g(a)\neq 0$).*

2. Continuity of a composite function. Assume that the function $y = \varphi(x)$ is defined on the set X and Y is the range of this function. Assume, furthermore, that a function $u = f(y)$ is defined on the set Y. Then we say that a *composite function* is defined on the set X and write $u = f(y)$, where $y = \varphi(x)$, or $u = f(\varphi(x))$.

Theorem 8. *Assume that the function $y = \varphi(x)$ is continuous at the point a and the function $u = f(y)$ is continuous at the point $b = \varphi(a)$. Then the composite function $u = f(\varphi(x)) = F(x)$ is continuous at the point a.*

3. Continuity of elementary functions.

The functions $y = C = \text{const}$, $y = x^\alpha$, $y = a^x$, $y = \log_a x$ $(a > 0, a \neq 1)$, $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = \cot x$, $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, $y = \operatorname{arccot} x$ are the *simplest* (or *basic*) *elementary functions.*

A function is called *elementary* if it can be obtained by means of a finite number of arithmetic operations and superpositions involving basic elementary functions.

The collection of all elementary functions constitutes a *class of elementary functions.*

Besides the basic elementary functions use is often made of the so-called hyperbolic functions:

hyperbolic sine ($\sinh x = (e^x - e^{-x})/2$),

hyperbolic cosine ($\cosh x = (e^x + e^{-x})/2$),

hyperbolic tangent ($\tanh x = \sinh x/\cosh x$),

hyperbolic cotangent ($\coth x = \cosh x/\sinh x$).

Theorem 9. *Any elementary function defined in the neighbourhood of a point is continuous at that point.*

4. The second remarkable limit

$$\lim_{x\to 0} (1 + x)^{1/x} = e \simeq 2.718281828459045 \ldots .$$

Note that this limit is an indeterminate form of type 1^∞.

5. Classification of points of discontinuity. Let a be a limit point of the domain of the function $f(x)$. The point a is a *point of discontinuity* of the function $f(x)$ if at this point $f(x)$ is not continuous. Let $f(x)$ be defined in a neighbourhood of the point a, except, maybe, the point a itself. Then a is

(1) a *point of removable discontinuity* of the function $f(x)$ if there is a limit $\lim\limits_{x\to a} f(x) = b$, but either $f(x)$ is not defined at the point a or $f(a) \neq b$ (if we set $f(a) = b$,

then the function $f(x)$ becomes continuous at the point a, i.e. the discontinuity will be removed),

(2) a *point of discontinuity of the first kind* of the function $f(x)$ if there are $f(a+0)$ and $f(a-0)$ but $f(a+0) \neq f(a-0)$,

(3) a *point of discontinuity of the second kind of the function* $f(x)$ if at least one of the one-sided limits of the function $f(x)$ does not exist at the point a.

II. Control Questions and Assignments

1. Formulate the definitions of (a) the continuity of a function at a point, (b) the continuity of a function on the right (on the left) of a point.

2. Formulate the necessary and sufficient conditions for the continuity of a function at a point.

3. Test the function $f(x)$ for continuity at an arbitrary point a:

(a) $f(x) = x^2$,

(b) $f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1 \text{ if } x \text{ is rational} \end{cases}$ (Dirichlet's function).

4. What functions are known as elementary?

5. Prove that the function $y = \sin x$ is continuous at any point a.

6. For what values of the argument x is the function $f(x) = \arcsin(\ln x)$ continuous? On the basis of what theorem?

7. What points are known as points of discontinuity of a function?

8. Give a definition of a point of removable discontinuity and points of discontinuity of the first and the second kind.

9. Find the points of discontinuity of the Dirichlet function. Indicate the kind of those points of discontinuity.

10. Indicate the kind of the point of discontinuity of the function $f(x)$: (a) $f(x) = \operatorname{sgn} x$, (b) $f(x) = |x|/x$.

11. Formulate the theorem on the continuity of a composite function. Using this theorem and the first remarkable limit, calculate $\lim\limits_{x\to 0} \ln \dfrac{\sin x}{x}$.

12. What functions are known as hyperbolic? Do they belong to the class of elementary functions? For what values of the argument are these functions continuous?

III. Worked Problems

1. Test the function $f(x)$ for continuity and indicate the kind of its points of discontinuity if

(a) $f(x)=\frac{x^2}{x}$, (b) $f(x)=e^{-1/x}$,

(c) $f(x)=\begin{cases} x & \text{for } x\leqslant 1, \\ \ln x & \text{for } x>1. \end{cases}$

$\triangle$ (a) For $x\neq 0$ the function $f(x)=x$ and is not defined for $x=0$. Since $\lim\limits_{x\to a} x=a$ $\forall a$, it follows that $\lim\limits_{x\to a} f(x)=a=f(a)$ for $a\neq 0$ and, consequently, $f(x)$ is continuous at any point $a\neq 0$. At the point $x=0$ the function $f(x)$ has a removable discontinuity since there is a limit $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0} x=0$.

(b) The function $f(x)=e^{-1/x}$ is elementary since it is a superposition of the functions $y=-x^{-1}$ and $f=e^y$. The function $f(x)$ is defined for all x, except for $x=0$. Consequently, according to Theorem 9, it is continuous at any point $x\neq 0$. Since $f(x)$ is defined in the neighbourhood of the point $x=0$ and is not defined at the point $x=0$ itself, it follows that $x=0$ is a point of discontinuity. Let us calculate $f(0+0)$ and $f(0-0)$ using Heine's definition of a one-sided limit of a function. We consider an arbitrary infinitesimal sequence $\{x_n\}$ such that $x_n>0$ $\forall n$. Since $\lim\limits_{n\to\infty}(-1/x_n)=-\infty$, we have $\lim\limits_{n\to\infty} e^{-1/x_n}=0$. Consequently, $\lim\limits_{x\to 0+0} e^{-1/x}=0$. Let us consider now an arbitrary infinitesimal sequence $\{x'_n\}$ such that $x'_n<0$ $\forall n$. Since $\lim\limits_{n\to\infty}(-1/x'_n)=+\infty$, it follows that $\lim\limits_{n\to\infty} e^{-1/x'_n}=+\infty$. Therefore $\lim\limits_{x\to 0-0} e^{-1/x}=+\infty$, i.e. $f(0-0)=+\infty$.

Thus the limit of $f(x)$ on the left of the point $x=0$ does not exist and, hence, $x=0$ is a point of discontinuity of the second kind.

(c) We shall prove the continuity of $f(x)$ at the point $a\neq 1$. We take $\varepsilon<|a-1|$, $\varepsilon>0$. Then the neighbourhood of the point a does not include the point $x=1$

if $\varepsilon < |a - 1|$. In this ε-neighbourhood $f(x)$ coincides either with the function $\varphi(x) = x$ if $a < 1$ or with the function $\psi(x) = \ln x$ if $a > 1$. Since these basic elementary functions are continuous at the point a, the function $f(x)$ is continuous at any point $a \neq 1$. We test the function $f(x)$ for continuity at the point $a = 1$. For that purpose, we calculate its one-sided limits at this point utilizing the continuity of the functions $\varphi(x)$ and $\psi(x)$ at the point $a = 1$ and Theorem 6. We obtain

$$f(1+0) = \lim_{x\to 1+0} \ln x = \lim_{x\to 1} \ln x = \ln 1 = 0,$$

$$f(1-0) = \lim_{x\to 1-0} x = \lim_{x\to 1} x = 1.$$

Thus $f(1+0) \neq f(1-0)$, and therefore $f(x)$ has a discontinuity of the first kind at the point $a = 1$. ▲

2. Prove that

(a) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$,

(b) $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \ln a$, $a > 0$, $a \neq 1$.

△ (a) We represent the function $\dfrac{\ln(1+x)}{x}$ in the form $\ln(1+x)^{1/x} = \ln y$, where $y = (1+x)^{1/x}$. Since $\lim\limits_{x\to 0}(1+x)^{1/x} = \mathrm{e}$ and the function $\ln y$ is continuous at the point $y = \mathrm{e}$, it follows that $\lim\limits_{x\to 0} \ln(1+x)^{1/x} = \ln \mathrm{e} = 1$.

(b) Let us consider the function $y = \varphi(x) = a^x - 1$. It is continuous at the point $x = 0$ and $y(0) = 0$. In this case

$$x = \log_a(1+y) \quad \text{and} \quad \frac{a^x - 1}{x} = \frac{y}{\log_a(1+y)}.$$

We calculate $\lim\limits_{y\to 0} \dfrac{y}{\log_a(1+y)}$ using the result obtained in item (a):

$$\lim_{y\to 0} \frac{y}{\log_a(1+y)} = \lim_{y\to 0} \frac{y \ln a}{\ln(1+y)}$$

$$= \ln a \frac{1}{\lim\limits_{y\to 0} \dfrac{\ln(1+y)}{y}} = \frac{\ln a}{1} = \ln a.$$

We shall consider now the function $f(y)$ continuous at the point $y=0$:

$$f(y)=\begin{cases} \dfrac{y}{\log_a(1+y)} & \text{for } y\neq 0, \\ \ln a & \text{for } y=0. \end{cases}$$

According to Theorem 8, the composite function

$$f(\varphi(x))=\begin{cases} \dfrac{a^x-1}{x} & \text{for } x\neq 0, \\ \ln a & \text{for } x=0 \end{cases}$$

is continuous at the point $x=0$. Therefore, $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$. ▲

IV. Problems and Exercises for Independent Work

15. Test the function $f(x)$ for continuity and indicate the kind of its points of discontinuity (see Exercises 1-4):

(a) $f(x)=x\sin(1/x)$, (b) $f(x)=\sin(1/x)$, (c) $f(x)=\{x\}$,

(d) $f(x)=\begin{cases} x^2 & \text{if } x \text{ is an irrational number}, \\ 1 & \text{if } x \text{ is a rational number}, \end{cases}$

(e) $f(x)=\dfrac{x^2+2}{x^3+1}$, (f) $f(x)=\arctan(1/x)$,

(g) $f(x)=\dfrac{1}{1-e^{x/(1-x)}}$, (h) $f(x)=\ln\dfrac{x^2}{(x+1)(x-3)}$,

(i) $f(x)=\begin{cases} x^2 & \text{for } 0\leqslant x<1, \\ 2-x & \text{for } 1<x\leqslant 2, \end{cases}$

(j) $f(x)=\begin{cases} x & \text{for } |x|\leqslant 1, \\ 1 & \text{for } |x|>1, \end{cases}$

(k) $f(x)\ \begin{cases} \cos(\pi x/2) & \text{for } |x|\leqslant 1, \\ |x-1| & \text{for } |x|>1. \end{cases}$

16. Prove that

(a) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$, (b) $\lim\limits_{x\to 0}\dfrac{(1+x)^a-1}{x}=a$,

(c) $\lim\limits_{x\to 0}\dfrac{\sinh x}{x}=1$, (d) $\lim\limits_{x\to 0}\dfrac{\tanh x}{x}=1$,

(e) $\lim\limits_{x\to 0}\dfrac{1-\cosh x}{x^2}=-\dfrac{1}{2}$.

3.3. Comparison of Infinitesimal Functions. The Order Symbol "o" and its Properties

I. Fundamental Concepts and Theorems

1. Comparison of infinitesimal functions. The function $\alpha(x)$ is *infinitesimal as $x \to a$* (at the point a) if $\lim_{x\to a} \alpha(x) = 0$. Let $\alpha(x)$ and $\beta(x)$ be two infinitesimal functions as $x \to a$. The functions $\alpha(x)$ and $\beta(x)$ are

(a) *infinitesimal functions of the same order as $x \to a$ (at the point a)* if

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = c \neq 0,$$

(b) *equivalent infinitesimal functions as $x \to a$ (at the point a)* if

$$\lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = 1 \text{ (the notation is } \alpha \sim \beta \text{ as } x \to a).$$

If $\lim_{x\to a} \frac{\alpha(k)}{\beta(x)} = 0$, then we say that $\alpha(x)$ is an *infinitesimal of a higher order of smallness as $x \to a$ (at the point a) than* $\beta(x)$ and write $\alpha = o(\beta)$ as $x \to a$ (α is equal to "o small" of β as $x \to a$).

For example, $x^2 = o(x)$ as $x \to 0$.

Similar definitions are true for the cases $x \to a + 0$, $x \to a - 0$ and $x \to \infty$.

It should be borne in mind that the equalities which contain "o" are conventional. For instance, the equality $x^2 = o(x)$ holds true as $x \to 0$ but the equality $o(x) = x^2$ does not since the symbol $o(x)$ denotes not a specific function but any function which, as $x \to 0$, is an infinitesimal of a higher order of smallness than x. There are infinitely many functions of this kind. In particular, any function x^p (where $p > 1$) is $o(x)$ as $x \to 0$. Thus the equality $x^2 = o(x)$, as $x \to 0$, means that the function x^2 belongs to the set of infinitesimal functions of a higher order of smallness, as $x \to 0$, than x. Therefore, the reverse of this equality, i.e. $o(x) = x^2$, does not hold true since the whole set of functions $o(x)$ does not reduce to a single function x^2.

2. The properties of the symbol "o"

Theorem 10. *Let $\alpha_1(x)$ and $\alpha_2(x)$ be two arbitrary infinitesimal functions as $x \to a$ such that* $\alpha_1(x) = o(\beta)$

and $\alpha_2(x) = o(\beta)$. *Then* $\alpha_1(x) + \alpha_2(x) = o(\beta)$ *as* $x \to a$.

This theorem can be written in the concise form $o(\beta) + o(\beta) = o(\beta)$.

Here are a number of other properties of the symbol "o" (everywhere we assume that $\alpha \to 0$ and $\beta \to 0$ as $x \to a$).

1°. $o(\beta) + o(\beta) = o(\beta)$.

2°. $o(\beta) - o(\beta) = o(\beta)$.

3°. $o(c\beta) = o(\beta)$ $\forall$ number $c \neq 0$.

4°. $co(\beta) = o(\beta)$ $\forall$ number $c \neq 0$.

5°. $o(\beta^n) = o(\beta^k)$, $n \geqslant 2$ $(n \in \mathrm{N})$, $k = 1, 2, \ldots, n-1$.

6°. $(o(\beta))^n = o(\beta^n)$ $\forall n \in \mathrm{N}$.

7°. $\beta^n o(\beta) = o(\beta^{n+1})$ $\forall n \in \mathrm{N}$.

8°. $\dfrac{o(\beta^n)}{\beta} = o(\beta^{n-1})$, $n \geqslant 2$ $(n \in \mathrm{N})$.

We use the symbol $o(1)$ for any infinitesimal function as $x \to a$. Then property 8° is also valid for $n = 1$, i.e. $\dfrac{o(\beta)}{\beta} = o(1)$.

9°. $o\left(\sum\limits_{k=1}^{n} c_k\beta^k\right) = o(\beta)$, where c_k are numbers.

10°. $o(o(\beta)) = o(\beta)$.

11°. $o(\beta + o(\beta)) = o(\beta)$.

12°. $\alpha\beta = o(\alpha)$ and $\alpha\beta = o(\beta)$.

13°. If $\alpha \sim \beta$, then $\alpha - \beta = o(\alpha)$ and $\alpha - \beta = o(\beta)$.

II. Control Questions and Assignments

1. Give the definition of an infinitesimal function (a) for $x \to a$, (b) for $x \to \infty$. Give examples of functions of this kind.

2. Formulate the definition and give examples of an infinitesimal function $\alpha(x)$

(a) of the same order as the function $\beta(x)$ at the point a,

(b) equivalent to the function $\beta(x)$ at the point a,

(c) of a higher order of smallness, as $x \to a$, than $\beta(x)$.

Interpret the symbolic notation $\alpha = o(\beta)$ as $x \to a$.

3. Give examples of functions $\alpha(x)$ for which the following equalities hold true: (a) $\alpha(x) = o(x)$ as $x \to 0$,

(b) $\alpha(x) = o(\sqrt{1-x})$ as $x \to 1-0$, (c) $\alpha(x) = o(1/x^2)$ as $x \to \infty$.

4. Prove that $x^3 = o(x^2)$ as $x \to 0$. Find whether the equality $x^3 = o(\beta)$ holds true as $x \to 0$ if (a) $\beta(x) = x$, (b) $\beta(x) = x^2 \sqrt{|x|}$, (c) $\beta(x) = x^3 \sqrt{|x|}$, (d) $\beta(x) = x^2 \sin x$.

5. Prove that $(x-1)^2 = o(x-1)$ as $x \to 1$. Find whether the equality $(x-1)^2 = o(\beta)$ holds true as $x \to 1$ if (a) $\beta(x) = (x-1)^3$, (b) $\beta(x) = \sin(x-1)^2$, (c) $\beta(x) = \frac{(x-1)^2}{\ln x}$.

6. Prove that $1/x^4 = o(1/x^3)$ as $x \to \infty$. Does the equality $1/x^4 = o(\beta)$ hold true as $x \to \infty$ if

(a) $\beta(x) = \frac{1}{x^k}$ $(k = 1, 2)$, (b) $\beta(x) = \frac{1}{x^5}$,

(c) $\beta(x) = \frac{1}{(x+1)^4}$, (d) $\beta(x) = \frac{1}{x^3 \sin x}$,

(e) $\beta(x) = \frac{1}{(x-1)^4 \arctan(1/x)}$?

7. Are the functions $\sin x$ and x equivalent infinitesimals as $x \to 0$? Prove that $\sin x - x = o(x)$ as $x \to 0$.

8. Using the properties of the symbol "o", write an equality of the form $\alpha(x) = o(x^k)$ as $x \to 0$ for the function $\alpha(x)$ if

$$\alpha(x) = o(x^2) + o(x^2), \quad \alpha(x) = o(x) - o(x),$$
$$\alpha(x) = 5o(x), \quad \alpha(x) = o(3x^2), \quad \alpha(x) = (o(x))^3,$$
$$\alpha(x) = xo(x), \quad \alpha(x) = \frac{o(x)^5}{x^2},$$
$$\alpha(x) = o(-x + 2x^2 + x^4), \quad \alpha(x) = o(o(x^2)),$$
$$\alpha(x) = o(x + o(x)).$$

9. Using the properties of the symbol "o", write an equality of the form $\alpha(x) = o(1/x^k)$ as $x \to \infty$ for the function $\alpha(x)$ if

$$\alpha(x) = o(1/x) - o(1/x), \quad \alpha(x) = 1000o(1/x),$$
$$\alpha(x) = o(1000/x), \quad \alpha(x) = (o(1/\sqrt{|x|}))^2,$$
$$\alpha(x) = x^2 o(1/x^3), \quad \alpha(x) = o\left(\frac{1}{x^2} - \frac{1}{x}\right),$$
$$\alpha(x) = o\left(o\left(\frac{1}{x^2}\right)\right), \quad \alpha(x) = o\left(\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right).$$

III. Worked Problems

1. Find whether the equality $\alpha(x) = o(x)$ holds true as $x \to 0$ if (a) $\alpha(x) = 2x^2$, (b) $\alpha(x) = 3x$, (c) $\alpha(x) = \sqrt{|x|}$, (d) $\alpha(x) = \frac{x}{\ln|x|}$, (e) $\alpha(x) = 1 - \cos x$.

△ (a) $2x^2 = o(x)$ since $\lim\limits_{x\to 0} \frac{2x^2}{x} = 0$.

(b) The equality $3x = o(x)$ does not hold true since $\lim\limits_{x\to 0} \frac{3x}{x} = 3 \neq 0$. The functions $3x$ and x are infinitesimals of the same order of smallness when $x \to 0$.

(c) $\sqrt{|x|} \neq o(x)$ since $\lim\limits_{x\to 0} \frac{\sqrt{|x|}}{x} = \infty$.

(d) $\frac{x}{\ln|x|} = o(x)$ since $\lim\limits_{x\to 0} \left(\frac{x}{\ln|x|} : x\right) = 0$.

(e) $1 - \cos x = o(x)$ since $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = \lim\limits_{x\to 0} \frac{2\sin^2(x/2)}{x} = \lim\limits_{x\to 0} \left[\frac{\sin(x/2)}{x/2}\right]^2 \frac{x}{2} = 1 \cdot 0 = 0$. ▲

2. Find whether the equality $\alpha(x) = o(x^2)$ holds true as $x \to 0$ if (a) $\alpha(x) = \sin^2 x$, (b) $\alpha(x) = x^3$, (c) $\alpha(x) = 1 - \cos x$.

△ (a) $\sin^2 x \neq o(x^2)$ since $\lim\limits_{x\to 0} \frac{\sin^2 x}{x^2} = \lim\limits_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = 1$. The functions $\sin^2 x$ and x^2 are equivalent infinitesimals at the point $x = 0$.

(b) $x^3 = o(x^2)$ since $\lim\limits_{x\to 0} \frac{x^3}{x^2} = 0$.

(c) $1 - \cos x \neq o(x^2)$ since $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$. The functions $1 - \cos x$ and x^2 are infinitesimals of the same order of smallness at the point $x = 0$. ▲

3. Using the limits

(a) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, (b) $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$,

(c) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$,

(d) $\lim\limits_{x\to 0} \frac{\sqrt[n]{1+x}-1}{x} = \frac{1}{n}$ (n is a natural number),

represent the functions $\sin x$, $\cos x$, $\ln(1+x)$, $\sqrt[n]{1+x}$ in the form

$$\psi(x) = a_0 + a_1 x^h + o(x^h) \quad \text{as} \quad x \to 0,$$

where $k = 1$ or 2, a_0 and a_1 are some numbers.

△ We shall first prove that if $\alpha(x)$ and $\beta(x)$ are infinitesimals of the same order of smallness when $x \to a$, i.e. $\lim \frac{\alpha(x)}{\beta(x)} = c \neq 0$, then $\alpha(x) = c\beta(x) + o(\beta)$ as $x \to a$.

Indeed, since

$$\lim_{x\to a}\left(\frac{\alpha(x)}{\beta(x)} - c\right) = 0 \quad \text{or} \quad \lim_{x\to a}\frac{\alpha(x) - c\beta(x)}{\beta(x)} = 0,$$

we find, according to the definition of the symbol $o(\beta)$, that $\alpha(x) - c\beta(x) = o(\beta)$, or

$$\alpha(x) = c\beta(x) + o(\beta) \quad \text{as} \quad x \to a. \tag{1}$$

Using formula (1), we find from relations (a)-(d) that

$$\sin x = x + o(x) \quad \text{for } x \to 0, \tag{2}$$

$$\cos x = 1 - \frac{1}{2}x^2 + o(x^2) \quad \text{for } x \to 0, \tag{3}$$

$$\ln(1+x) = x + o(x) \quad \text{for } x \to 0, \tag{4}$$

$$\sqrt[n]{1+x} = 1 + \frac{1}{n}x + o(x) \quad \text{for } x \to 0. \tag{5}$$

Formulas (2)-(5) are *asymptotic formulas*, or *asymptotic expansions*, or, else, *asymptotic representations of the functions* $\sin x$, $\cos x$, $\ln(1+x)$, $\sqrt[n]{1+x}$ as $x \to 0$. The least term on the right-hand side of these formulas [$o(x)$ or $o(x^2)$] is the *remainder of an asymptotic formula*. ▲

4. Prove properties 2°, 3°, 6°, 9° and 10° of the symbol "o".

△ Recall that the symbol $o(\beta)$, appearing on the left-hand side of the formulas, means any infinitesimal function at the point a of a higher order of smallness than $\beta(x)$.

1. We shall first prove properties 2°, 3° and 6°, i.e.

$$o(\beta) - o(\beta) = o(\beta), \tag{6}$$

$$o(c\beta) = o(\beta) \quad \forall \text{ number } c \neq 0, \tag{7}$$

$$(o(\beta))^n = o(\beta^n) \quad \forall n \in \mathbb{N}. \tag{8}$$

We designate as $\alpha_1(x)$, $\alpha_2(x)$ and $\alpha(x)$ arbitrary infinitesimal functions at the point a such that $\alpha_1(x) = o(\beta)$, $\alpha_2(x) = o(\beta)$, $\alpha(x) = o(c\beta)$ as $x \to a$. By the definition of the symbol "o", these equalities mean that

$$\lim_{x\to a} \frac{\alpha_1(x)}{\beta(x)} = 0, \tag{9}$$

$$\lim_{x\to a} \frac{\alpha_2(x)}{\beta(x)} = 0, \tag{10}$$

$$\lim_{x\to a} \frac{\alpha(x)}{c\beta(x)} = 0 \quad (c \neq 0). \tag{11}$$

To prove the validity of relations (6)-(8), we must prove that

$$\alpha_1(x) - \alpha_2(x) = o(\beta), \quad \alpha(x) = o(\beta),$$

$$(\alpha_1(x))^n = o(\beta^n), \text{ i.e.}$$

$$L_1 = \lim_{x\to a} \frac{\alpha_1(x) - \alpha_2(x)}{\beta(x)} = 0,$$

$$L_2 = \lim_{x\to a} \frac{\alpha(x)}{\beta(x)} = 0, \quad L_3 = \lim_{x\to a} \frac{(\alpha_1(x))^n}{(\beta(x))^n} = 0.$$

Taking (9) and (10) into account, we obtain

$$L_1 = \lim_{x\to a} \frac{\alpha_1(x)}{\beta(x)} - \lim_{x\to a} \frac{\alpha_2(x)}{\beta(x)} = 0 - 0 = 0,$$

$$L_3 = \left(\lim_{x\to a} \frac{\alpha_1(x)}{\beta(x)}\right)^n = 0^n = 0.$$

Using relation (11), we find that

$$L_2 = c \lim_{x\to a} \frac{\alpha(x)}{c\beta(x)} = c \cdot 0 = 0.$$

We have thus proved the validity of formulas (6)-(8).

2. We shall now prove properties 9^0 and 10^0, i.e.

$$o\left(\sum_{k=1}^{n} c_k \beta^k\right) = o(\beta) \quad (c_k \text{ are numbers}), \tag{12}$$

$$o(o(\beta)) = o(\beta). \tag{13}$$

We designate as $\psi(x)$, $\alpha(x)$ and $\varphi(x)$ arbitrary infinitesimal functions at the point a such that

$$\psi(x) = o\sum_{k=1}^{n} c_k\beta^k, \quad \alpha(x) = o(\beta), \quad \varphi(x) = o(\alpha) = o(o(\beta)),$$

i.e.

$$\lim_{x\to a}\frac{\psi(x)}{\sum\limits_{k=1}^{n} c_k\beta_k}=0, \tag{14}$$

$$\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=0, \tag{15}$$

$$\lim_{x\to a}\frac{\varphi(x)}{\alpha(x)}=0. \tag{16}$$

To prove the validity of relations (12) and (13), we must find that

$$L_1=\lim_{x\to a}\frac{\psi(x)}{\beta(x)}=0, \quad L_2=\lim_{x\to a}\frac{\varphi(x)}{\beta(x)}=0.$$

Taking (14), (15) and (16) into account, we obtain

$$L_1=\lim_{x\to a}\frac{\psi(x)}{\sum\limits_{k=1}^{n} c_k\beta^k}\lim_{x\to a}\frac{\sum\limits_{k=1}^{n} c_k\beta^k}{\beta(x)}=0\cdot c_1=0,$$

$$L_2=\lim_{x\to a}\frac{\varphi(x)}{\alpha(x)}\lim_{x\to a}\frac{\alpha(x)}{\beta(x)}=0\cdot 0=0.$$

We have thus proved the validity of relations (12) and (13). ▲

IV. Problems and Exercises for Independent Work

17. Using the limits

(a) $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a\ (a>0,\ a\neq 1)$, (b) $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$,

(c) $\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha$, (d) $\lim\limits_{x\to 0}\frac{\sinh x}{x}=1$,

(e) $\lim\limits_{x\to 0}\frac{\tanh x}{x}=1$, (f) $\lim\limits_{x\to 0}\frac{1-\cosh x}{x^2}=-\frac{1}{2}$,

derive the asymptotic formulas (as $x\to 0$) for the functions a^x, e^x, $(1+x)^\alpha$, $\sinh x$, $\tanh x$, $\cosh x$.

18. Prove properties 1^0, 4^0, 5^0, 7^0, 8^0, 11^0-13^0 of the symbol "o".

19. Find out whether the equality $o(o(x))=o(x^{1+\varepsilon})$ holds true as $x\to 0$ if (a) $\varepsilon>0$, (b) $\varepsilon=0$, (c) $-1<\varepsilon<0$. Substantiate the answer.

20. Find out whether each of the following equalities holds true: (a) $o(x + x^2) = o(x^2)$ as $x \to 0$, (b) $o(x) = o(x^2)$ as $x \to 0$, (c) $o(x^2) = o(x)$ as $x \to 0$, (d) $o(1/x) = o(1/x^2)$ as $x \to \infty$, (e) $o(1/x^2) = o(1/x)$ as $x \to \infty$. Substantiate the answer.

21. Using the properties of the symbol "o", write an equality of the form $\alpha(x) = o(1)$ or $\alpha(x) = o((x-a)^k)$ as $x \to a$ (k is a natural number) for the function $\alpha(x)$ if

(a) $\alpha(x) = o(-5x + x^2 - x^3 + o(-5x + x^2 - x^3))$ as $x \to 0$,

(b) $\alpha(x) = (x-1)\, o((x-1)^2 + o(x-1))$ for $x \to 1$,

(c) $\alpha(x) = \dfrac{1}{5x}\, o(3x + x^2)$ for $x \to 0$.

22. Using the properties of the symbol "o", write an equality of the form $\alpha(x) = o(1)$ or $\alpha(x) = o(1/x^k)$ as $x \to \infty$ (k is a natural number) for the function $\alpha(x)$ if

(a) $\alpha(x) = o\left(\dfrac{1}{2x^2} - \dfrac{1}{x} + o\left(\dfrac{1}{x}\right)\right)$,

(b) $\alpha(x) = \dfrac{2}{x^3} - \dfrac{1}{x^2}$, (c) $\alpha(x) = x^2 \cdot o\left(\dfrac{1}{x^3} + o\left(\dfrac{1}{x^3}\right)\right)$,

(d) $\alpha(x) = x\left(o\left(\dfrac{1}{x^2}\right) - o\left(\dfrac{1}{x^2}\right)\right)$,

(e) $\alpha(x) = 5x \cdot o\left(\dfrac{1}{x^2} + o\left(\dfrac{1}{x}\right)\right)$.

3.4. Calculating the Limits of Functions by Means of Asymptotic Formulas. Calculating the Limits of Exponential Power Functions

I. Fundamental Concepts and Theorems

1. Asymptotic formulas. In the examples and problems presented in 3.3 we obtained asymptotic formulas for the basic elementary functions as $x \to 0$. We shall tabulate these formulas:

I. $\sin x = x + o(x)$.

II. $\cos x = 1 - \dfrac{x^2}{2} + o(x^2)$.

III. $\ln(1 + x) = x + o(x)$.

IV. $a^x = 1 + x \ln a + o(x) \quad (a > 0)$;
$e^x = 1 + x + o(x)$.

V. $(1+x)^a = 1 + ax + o(x)$.
VI. $\tan x = x + o(x)$.
VII. $\sinh x = x + o(x)$.
VIII. $\cosh x = 1 + \frac{1}{2}x^2 + o(x^2)$.
IX. $\tanh x = x + o(x)$.

These formulas remain valid if the argument x in them is replaced either by x_n, where $\{x_n\}$ is an infinitely small sequence, or by $y(x)$, where $\lim_{x \to a} y(x) = 0$. For instance, the following representation resulting from formula (1) holds true:

$$\sin \frac{1}{n^2} = \frac{1}{n^2} + o\left(\frac{1}{n^2}\right),$$

where $\{o(1/n^2)\}$ is an infinitesimal sequence of a higher order than $\{1/n^2\}$, i.e. $\lim_{n\to\infty} \frac{o(1/n^2)}{1/n^2} = \lim_{n\to\infty} n^2 o(1/n^2) = 0$.

The function $y(x) = x - 1$ is infinitesimal as $x \to 1$, and therefore from formula III we get a relation

$$\ln(1 + y(x)) = y(x) + o(y) \quad \text{as} \quad x \to 1,$$

or

$$\ln(1 + (x - 1)) = \ln x = x - 1 + o(x - 1)$$

as $x \to 1$.

Using this relation and formula II, we write the asymptotic representation of the function $\cos \ln x$ as $x \to 1$:

$$\cos \ln x = \cos(x - 1 + o(x - 1))$$
$$= 1 - \frac{(x - 1 + o(x-1))^2}{2} + o((x - 1 + o(x - 1))^2).$$

On the basis of the properties of the symbol "o", we get

$$\frac{(x - 1 + o(x-1))^2}{2}$$
$$= \frac{(x-1)^2}{2} + (x - 1)\,o(x - 1) + \frac{1}{2}[o(x - 1)]^2$$
$$= \frac{(x-1)^2}{2} + o(x - 1)^2 + o(x - 1)^2 = \frac{(x-1)^2}{2} + o(x - 1)^2.$$

Similarly,

$$(x - 1 + o(x - 1))^2 = (x - 1)^2 + o(x - 1)^2.$$

By virtue of property 11° we have

$$o((x - 1)^2 + o(x - 1)^2) = o(x - 1)^2.$$

The final result is

$$\cos \ln x = 1 - \frac{(x-1)^2}{2} + o(x-1)^2 \text{ as } x \to 1.$$

2. Calculating the limits of exponential power functions. Consider the calculation of the limit, as $x \to a$, of the exponential power function $[u(x)]^{v(x)}$, where the functions $u(x)$ and $v(x)$ are defined in a neighbourhood of the point a, and $u(x) > 0$.

The following cases are possible here:

1. If $\lim\limits_{x\to a} u(x) = b > 0$, $\lim\limits_{x\to a} v(x) = c$, then $\lim\limits_{x\to a} u(x)^{v(x)} = b^c$.

2. If $\lim\limits_{x\to a} u(x) = 0$, $\lim\limits_{x\to a} v(x) = c > 0$ (or $+\infty$), then $\lim\limits_{x\to a} u^v = 0$.

3. If $\lim\limits_{x\to a} u(x) = 0$, $\lim\limits_{x\to a} v(x) = c < 0$ (or $-\infty$), then $\lim\limits_{x\to a} u^v = +\infty$.

4. If $\lim\limits_{x\to a} u(x) = 0$, $\lim\limits_{x\to a} v(x) = 0$, then $\lim\limits_{x\to a} u(x)^{v(x)}$ is an indeterminate form of type 0^0.

5. If $\lim\limits_{x\to a} u(x) = +\infty$, $\lim\limits_{x\to a} v(x) = c > 0$ (or $+\infty$), then $\lim\limits_{x\to a} u^v = +\infty$.

6. If $\lim\limits_{x\to a} u(x) = +\infty$, $\lim\limits_{x\to a} v(x) = c < 0$, (or $-\infty$), then $\lim\limits_{x\to a} u^v = 0$.

7. If $\lim\limits_{x\to a} u(x) = +\infty$, $\lim\limits_{x\to a} v(x) = 0$, then $\lim\limits_{x\to a} u^v$ is an indeterminate form of type ∞^0.

8. If $\lim\limits_{x\to a} u(x) = 1$, $\lim\limits_{x\to a} v(x) = \infty$, then $\lim\limits_{x\to a} u^v$ is an indeterminate form of type 1^∞. The second remarkable limit is an example of an indeterminate form of this type:

$$\lim_{x\to 0} (1+x)^{1/x} = e.$$

If we represent u^v as $e^{v \ln u}$, then each of the indeterminate forms (0^0, ∞^0, 1^∞) reduces to an indeterminate form of type $0\cdot\infty$ for the function $v \ln u$.

If, in this case, $\lim\limits_{x\to a} v \ln u = b$, then $\lim\limits_{x\to a} u^v = e^b$.

II. Control Questions and Assignments

1. Write asymptotic formulas for the functions $\sin x$, $\tan x$, $\cos x$, $\ln(1+x)$, e^x, a^x, $(1+x)^a$, $\sinh x$, $\tanh x$, $\cosh x$ as $x \to 0$.

2. Write asymptotic formulas with the remainder of the form $o(x^\alpha)$ as $x \to 0$ or $o(1/x^\alpha)$ as $x \to \infty$ $(\alpha > 0)$ for the composite functions $\sin y$, $\tan y$, $\cos y$, $\ln(1+y)$, e^y, a^y, $(1+y)^a$, $\sinh y$, $\tanh y$, $\cosh y$ if (a) $y = 3x$ and $x \to 0$, (b) $y = \sqrt{x}$ and $x \to +0$, (c) $y = x^3$ and $x \to 0$, (d) $y = 1/x$ and $x \to \infty$.

3. Write asymptotic formulas with the remainder of the form $o(1/n^\alpha)$ $(\alpha > 0)$ for the sequences $\sin x_n$, $\tan x_n$, $\cos x_n$, $\ln(1+x_n)$, e^{x_n}, a^{x_n}, $(1+x_n)^a$, $\sinh x_n$, $\tanh x_n$, $\cosh x_n$ if (a) $x_n = 1/n$, (b) $x_n = 1/n^2$, (c) $x_n = \ln\left(1+\frac{1}{n}\right)$, (d) $x_n = e^{1/n} - 1$.

4. Give a definition of an infinitesimal sequence $\{\alpha_n\}$ of a higher order than $\{1/n\}$ as $n \to \infty$. Give a definition of the infinitely small function $\alpha(x)$ of a higher order than $1/x$ as $x \to \infty$. Write the symbolic notations for α_n and $\alpha(x)$.

5. Of what order of smallness are the sequence $\alpha_n = n\,o(1/n)^2$ as $n \to \infty$ and the function $\alpha(x) = x\,o(1/x^2)$ as $x \to \infty$ as compared to $\{1/n\}$ and $\beta(x) = 1/x$ respectively?

6. Using the asymptotic formula IV and Heine's definition of the limit of a function, prove that $e^{1/n} = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right)$ as $n \to \infty$.

7. Find whether the following equalities hold true:

$$\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 + o(x^2) \quad \text{as } x \to 0,$$

$$\cosh\frac{\pi}{n} = 1 + \frac{1}{2}\frac{\pi^2}{n^2} + o\left(\frac{1}{n^2}\right) \quad \text{as } n \to \infty.$$

Substantiate the answer.

8. Enumerate possible cases for calculating the limits of exponential power functions. Give examples of three types of indeterminate forms for those functions.

9. Calculate the following limits:

(a) $\lim\limits_{x\to\infty}\left(\frac{x-2}{2x-3}\right)^x$, (b) $\lim\limits_{x\to\infty}\left(\frac{x-2}{x+3}\right)^{2x}$, (c) $\lim\limits_{x\to+0}(x)^{1/\ln x}$.

III. Worked Problems

1. Calculate $\lim\limits_{x\to 0}\dfrac{\sin\sin\tan(x^2/2)}{\ln\cos 3x}$.

△ We write the asymptotic expansion of the numerator using the asymptotic formulas for the sine and tangent and the properties of the symbol "o":

$$\begin{aligned}\sin\sin\tan\frac{x^2}{2}&=\sin\sin\left(\frac{x^2}{2}+o\left(\frac{x^2}{2}\right)\right)\\&=\sin\left[\frac{x^2}{2}+o\left(\frac{x^2}{2}\right)+o\left(\frac{x^2}{2}+o\left(\frac{x^2}{2}\right)\right)\right]\\&=\sin\left(\frac{x^2}{2}+o(x^2)+o(x^2)\right)\\&=\sin\left(\frac{x^2}{2}+o(x^2)\right)=\frac{x^2}{2}+o(x^2).\end{aligned}$$

We have used here the fact that $o\left(\frac{x^2}{2}+o\left(\frac{x^2}{2}\right)\right)=o(x^2)$ and $o(x^2)+o(x^2)=o(x^2)$.

Now we shall derive the asymptotic expansion of the denominator using the asymptotic formulas for the cosine and logarithm:

$$\begin{aligned}\ln\cos 3x&=\ln\left(1-\frac{(3x)^2}{2}+o((3x)^2)\right)\\&=\ln\left(1+\left(-\frac{9x^2}{2}+o(x^2)\right)\right)\\&=\left(-\frac{9x^2}{2}+o(x^2)\right)+o\left(-\frac{9x^2}{2}+o(x^2)\right)\\&=-\frac{9x^2}{2}+o(x^2)+o(x^2)=-\frac{9x^2}{2}+o(x^2).\end{aligned}$$

We have used here the fact that

$$o((3x)^2)=o(x^2),\quad o\left(-\frac{9x^2}{2}+o(x^2)\right)=o(x^2),$$
$$o(x^2)+o(x^2)=o(x^2).$$

Thus the limit is equal to

$$\lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)}{-\frac{9x^2}{2}+o(x^2)}=\lim_{x\to 0}\frac{\frac{1}{2}+\frac{o(x^2)}{x^2}}{-\frac{9}{2}+\frac{o(x^2)}{x^2}}$$
$$=\frac{\frac{1}{2}+\lim\limits_{x\to 0}\frac{o(x^2)}{x^2}}{-\frac{9}{2}+\lim\limits_{x\to 0}\frac{o(x^2)}{x^2}}=-\frac{1}{9}.$$

Here we have used the fact that according to the definition of the symbol "o" $\lim\limits_{x\to 0}\frac{o(x^2)}{x^2}=0$. ▲

2. Calculate $\lim\limits_{x\to a}\frac{a^x-a^a}{x-a}$ $(a>0)$.

△ We set $y=x-a$, and then we can write the limit as

$$\lim_{y\to 0}\frac{a^{y+a}-a^a}{y}=a^a\lim_{y\to 0}\frac{a^y-1}{y}.$$

Since $a^y=1+y\ln a+o(y)$, we have

$$a^a\lim_{y\to 0}\frac{a^y-1}{y}=a^a\lim_{y\to 0}\frac{y\ln a+o(y)}{y}$$
$$=a^a\left(\ln a+\lim_{y\to 0}\frac{o(y)}{y}\right)=a^a\ln a.$$

Thus, $\lim\limits_{x\to a}\frac{a^x-a^a}{x-a}=a^a\ln a$. ▲

3. Calculate $\lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})$.

△ Using the asymptotic formula V for $x=1/n^2$ and $a=1/2$, we obtain

$$\sqrt{n^2+1}=n\left(1+\frac{1}{n^2}\right)^{1/2}=n\left(1+\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)$$
$$=n+\frac{1}{2n}+o\left(\frac{1}{n}\right).$$

Hence

$$\sin(\pi\sqrt{n^2+1})=\sin\left(\pi n+\frac{\pi}{2n}+o\left(\frac{1}{n}\right)\right)$$
$$=(-1)^n\sin\left(\frac{\pi}{2n}+o\left(\frac{1}{n}\right)\right).$$

The sequence $\{(-1)^n\}$ is bounded and $\left\{\sin\left(\frac{\pi}{2n}+o\left(\frac{1}{n}\right)\right)\right\}$ is infinitesimal, and therefore the product of these two sequences is an infinitely small sequence. Thus $\lim\limits_{n\to\infty}\sin(\pi\sqrt{n^2+1})=0$. ▲

4. Calculate $\lim\limits_{x\to 0}(\cos x)^{\cot^2 x}$.

△ This limit is an indeterminate form of type 1^∞ since $\lim\limits_{x\to 0}\cos x=1$, $\lim\limits_{x\to 0}\cot^2 x=\infty$. We write $(\cos x)^{\cot^2 x}$

in the form $e^{\cot^2 x \cdot \ln \cos x}$ and calculate $L = \lim\limits_{x\to 0} \cot^2 x \cdot \ln \cos x$. For that purpose, we write an asymptotic expansion for $\ln \cos x$ and $\sin^2 x$ as $x \to 0$:

$$\ln \cos x = \ln\left(1 - \frac{x^2}{2} + o(x^2)\right) = -\frac{x^2}{2} + o(x^2),$$
$$\sin^2 x = (x + o(x))^2 = x^2 + o(x^2).$$

Using these relations, we find that

$$L = \lim_{x\to 0} \cos^2 x \cdot \lim_{x\to 0} \frac{-\frac{x^2}{2} + o(x^2)}{x^2 + o(x^2)} = -\frac{1}{2}.$$

Thus the required limit is equal to $e^L = e^{-1/2}$. ▲

5. Calculate $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right)^{\tan(1/n)}$.

△ This limit is an indeterminate form of type 0^0. To calculate it, we write $\left(\frac{1}{n}\right)^{\tan(1/n)}$ in the form $e^{\tan(1/n)\cdot \ln(1/n)}$ and calculate $L = \lim\limits_{n\to\infty} \left(\tan \frac{1}{n} \times \ln \frac{1}{n}\right)$. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \left[-\ln n \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right)\right] \\ &= -\lim_{n\to\infty} \frac{\ln n}{n} - \lim_{n\to\infty} \left[\ln n \cdot o\left(\frac{1}{n}\right)\right] \\ &= 0 - \lim_{n\to\infty} \frac{\ln n}{n} \cdot \lim_{n\to\infty} \frac{o\left(\frac{1}{n}\right)}{\frac{1}{n}} = 0 \end{aligned}$$

since $\lim\limits_{n\to\infty} \frac{\ln n}{n} = 0$ (see 2.4) and $\lim\limits_{n\to\infty} \frac{o(1/n)}{1/n} = 0$.

Thus $L = 0$, i.e. the required limit is equal to 1. ▲

IV. Problems and Exercises for Independent Work

23. Write the asymptotic expansion for the following functions, as $x \to 0$, with the remainder of the form $o(x^\alpha)$, where $\alpha \geqslant 0$:

(a) $\sin^2(5\sqrt{x})$, $\sin^2(5\sqrt{x} + x)$ $(x > 0)$, (b) $\cos(4x^2)$, $\cos(4x^2 + x)$,

(c) e^{2x}, $e^{2x+\sqrt{x}}$ $(x>0)$, (d) $\ln(1-x^2)$, $\ln(1-x^2+x)$,

(e) $3-\sqrt[3]{27-x}$, $3-\sqrt[3]{27-x+\sqrt{x}}$ $(x>0)$, (f) 2^{x^3}, $2^{x^3+x^2}$,

(g) $\ln\cos 2x$, $\ln\cos(2x+x^2)$, (h) $\cos\sqrt{\sin x}$, $\cosh\sqrt{\sin x}$ $(x>0)$,

(i) $\ln(e^x+\sqrt{x})$ $(x>0)$, (j) $5^{e^x-\cos\sqrt{|x|}}$,

(k) $\sqrt[3]{\cos\sqrt{x}}$ $(x>0)$, (l) $\cos x\cos x^2-1$.

24. Write the asymptotic expansions for the functions, as $x\to 2$, with the remainder of the form $o(x-2)^\alpha$, where $\alpha\geqslant 0$: (a) $\sin(x-2)^2$, (b) $(3-x)^\beta$, (c) $\ln(x-1)$, (d) $\cos(\pi x)$, (e) $\tan(\pi x^2)$, (f) $\sqrt[5]{x-1}-\sqrt[7]{x-1}$, (g) x^x-4.

25. Write asymptotic expansions for the functions, as $x\to\infty$, with the remainder of the form $o(1/x^\alpha)$, where $\alpha\geqslant 0$:

(a) $\sqrt{x^2+x}-x$, (b) $\sqrt[3]{x^3+x}-x$,

(c) $\sqrt{(x+1)(x+2)(x+3)(x+4)}-x$,

(d) $\sin(1/\sqrt[3]{x^2})$, $\sinh(1/\sqrt[3]{x})$, (e) $\cos(1/x^2)$,

(f) $5^{1/x}$, (g) $\ln\cos(2/x)$, $\ln\cosh(2/x)$, (h) $e^{1/\sqrt{x}}-1$ $(x>0)$.

26. Write asymptotic formulas for sequences with the remainder of the form $o(1/n^\alpha)$, where $\alpha\geqslant 0$: (a) $\sqrt[3]{n^3+n^2}-n$, (b) $2^{1/n}+7^{1/n}-2$, (c) $\sin(1/\sqrt{n})$.

27. Calculate each of the following limits:

(a) $\lim\limits_{x\to 0}\dfrac{\cos x-\cos 3x}{x^2}$, (b) $\lim\limits_{x\to\pi/3}\dfrac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}$,

(c) $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$,

(d) $\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+ax}-\sqrt[n]{1+bx}}{x}$ $(m, n\in\mathbb{N})$,

(e) $\lim\limits_{x\to+\infty}\left(\sqrt{1+x+x^2}-\sqrt{1-x+x^2}\right)$,

(f) $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$, (g) $\lim\limits_{x\to a}\dfrac{\ln x-\ln a}{x-a}$,

(h) $\lim\left(\dfrac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}$ as $x\to+0$, $x\to 1$, $x\to+\infty$,

(i) $\lim\limits_{x\to\pi/4+0}\left[\tan\left(\frac{\pi}{8}+x\right)\right]^{\tan 2x}$,

(j) $\lim\limits_{x\to+\infty}\left(\frac{ax+1}{bx+2}\right)^{x}$ $(a^2+b^2\neq 0)$,

(k) $\lim\limits_{x\to\pi/2}(\sin x)^{\tan x}$, (l) $\lim\limits_{n\to\infty} n^{(\cosh(\pi/n)-1)}$,

(m) $\lim\limits_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^{n}$ $(a>0,\ b>0)$.

28. Calculate each of the following limits:

(a) $\lim\limits_{x\to 0}\frac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1}{x}$ $(m,\ n\in\mathrm{N})$,

(b) $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+2x}-\sqrt[4]{1+9x}}{1-\sqrt{1-\frac{x}{2}}}$, (c) $\lim\limits_{x\to+0}\frac{1-\sqrt{\cos x}}{\sin^4 3\sqrt{x}}$,

(d) $\lim\limits_{x\to 0}\frac{\ln\cosh 2x}{\ln\cos 3x}$, (e) $\lim\limits_{x\to 0}\frac{e^{x^2}-e^{3x}}{\sin(x^2/2)-\sin x}$,

(f) $\lim\limits_{n\to\infty} n\left(\sqrt[n]{a}-1\right)$ $(a>0)$, (g) $\lim\limits_{n\to\infty}\left(\frac{\cosh(\pi/n)}{\cos(\pi/n)}\right)^{n^2}$,

(h) $\lim\limits_{x\to\infty}\left(\frac{x+2}{2x-1}\right)^{x^2}$, (i) $\lim\limits_{x\to\infty}\left(\frac{3x^2+1}{2x^2-1}\right)^{\frac{x^3}{1-x}}$,

(j) $\lim\limits_{n\to\infty}\sin^n\left(\frac{2\pi n}{3n+1}\right)$, (k) $\lim\limits_{n\to\infty}\cos^n\frac{x}{\sqrt{n}}$,

(l) $\lim\limits_{x\to\pi/4}(\tan x)^{\tan 2x}$.

29. Calculate each of the following limits:

(a) $\lim\limits_{x\to 1}\frac{\sin^2(\pi\cdot 2^x)}{\ln\cos(\pi\cdot 2^x)}$, (b) $\lim\limits_{x\to a}\frac{x^x-a^a}{x-a}$ $(a>0)$,

(c) $\lim\limits_{x\to 0}\frac{\ln\tan\left(\frac{\pi}{4}+ax\right)}{\sin bx}$,

(d) $\lim\limits_{h\to 0}\frac{a^{x+h}+a^{x-h}-2a^x}{h^2}$ $(a>0)$,

(e) $\lim\limits_{x\to 0}\frac{\cos(xe^x)-\cos(xe^{-x})}{x^3}$, (f) $\lim\limits_{n\to\infty}\left(\cosh\frac{\pi}{n}-1\right)^{1/\ln n^{-1}}$,

(g) $\lim\limits_{n\to\infty} n^2\sin\left(\ln\sqrt{\cos\frac{\pi}{n}}\right)$,

(h) $\lim\limits_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{a}+\sin\frac{1}{n}\right]^{n}$,

(i) $\lim\limits_{n\to\infty} \cos(\pi\sqrt{n^2+n})$, (j) $\lim\limits_{x\to+\infty} (x-\ln\cosh x)$.

30. Calculate each of the following limits:

(a) $\lim\limits_{x\to+\infty} (\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x})$,

(b) $\lim\limits_{x\to\infty} x^{1/3}[(x+1)^{2/3}-(x-1)^{2/3}]$,

(c) $\lim\limits_{x\to 0} \dfrac{1-\cos x\cos 2x\cos 3x}{\sin^2 2x}$,

(d) $\lim\limits_{x\to 1} (1-x)\log_x 2$, (e) $\lim\limits_{x\to a} \dfrac{a^x-x^a}{x-a}$ $(a>0)$,

(f) $\lim\limits_{n\to\infty} n^2(\sqrt[n]{a}-\sqrt[n+1]{a})$ $(a>0)$,

(g) $\lim\limits_{x\to+0} \dfrac{\ln(e^{x^2}+2\sqrt{x})}{\tan\sqrt{x}}$, (h) $\lim\limits_{x\to 1} \dfrac{\sqrt[13]{x}-\sqrt[7]{x}}{\sqrt[5]{x}-\sqrt[3]{x}}$,

(i) $\lim\limits_{x\to\infty} \left[\cos\left(2\pi\left(\dfrac{x}{x+1}\right)^a\right)\right]^{x^2}$,

(j) $\lim\limits_{n\to\infty} \tan^n\left[\dfrac{\pi-4}{4}+\left(1+\dfrac{1}{n}\right)^{\alpha}\right]$,

(k) $\lim\limits_{x\to\infty} \left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$,

(l) $\lim\limits_{x\to+\infty} \dfrac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$, (m) $\lim\limits_{x\to-\infty} \dfrac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$.

Chapter 4

Derivatives and Differentials

4.1. The Derivative of a Function. Differentiation Rules

I. Fundamental Concepts and Theorems

1. Definition of the derivative. Let the function $y = f(x)$ be defined in a neighbourhood of the point x_0. The *increment* of this function at x_0 is a function of the argument Δx:

$$\Delta y = f(x_0+\Delta x) - f(x_0).$$

The *divided difference* $\dfrac{\Delta y}{\Delta x}$ is also a function of the argument Δx.

Definition. *The derivative of the function* $y = f(x)$ *at the point* x_0 *is the limit* $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ *(provided that it exists).*

The derivative of the function $y = f(x)$ at the point x_0 is designated as $f'(x_0)$ or $y'(x_0)$. The operation of finding the derivative is *differentiation*.

2. Table of derivatives of basic elementary functions

I. $(x^\alpha)' = \alpha x^{\alpha-1}$ (α is any number).

II. $(\sin x)' = \cos x$.

III. $(\cos x)' = -\sin x$.

IV. $(\log_a x)' = \frac{1}{x \ln a}$, in particular, $(\ln x)' = \frac{1}{x}$ $(x > 0)$,

V. $(a^x)' = a^x \ln a$, in particular, $(e^x)' = e^x$.

VI. $(\tan x)' = \frac{1}{\cos^2 x}$ $\left(x \neq \frac{\pi}{2} + \pi n,\ n \in \mathbf{Z}\right)$.

VII. $(\cot x)' = -\frac{1}{\sin^2 x}$ $(x \neq \pi n,\ n \in \mathbf{Z})$.

VIII. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$.

IX. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ $(-1 < x < 1)$.

X. $(\arctan x)' = \frac{1}{1+x^2}$.

XI. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

XII. $(\sinh x)' = \cosh x$.

XIII. $(\cosh x)' = \sinh x$.

XIV. $(\tanh x)' = \frac{1}{\cosh^2 x}$.

XV. $(\coth x)' = -\frac{1}{\sinh^2 x}$ $(x \neq 0)$.

3. Physical meaning of the derivative. The derivative $f'(x_0)$ is the rate of variation of the function $y = f(x)$ at the point x_0 (in other words, the rate of variation of the dependent variable y relative to the variation of the independent variable x at x_0). In particular, if x is time, $y = f(x)$ is the coordinate of the point, which moves along a straight line, at the moment x, then $f'(x_0)$ is the instantaneous velocity of the point at the moment x_0.

4. Geometrical meaning of the derivative. Consider the graph of the function $y = f(x)$ (Fig. 1). The points M and N have the following coordinates: $M(x_0, f(x_0))$,

$N(x_0 + \Delta x, f(x_0 + \Delta x))$. We shall designate the angle between the secant MN and the x-axis as $\varphi(\Delta x)$.

Definition. *If there is a limit* $\lim\limits_{\Delta x \to 0} \varphi(\Delta x) = \varphi_0$, *then the straight line* l, *with the slope* $k = \tan \varphi_0$, *which*

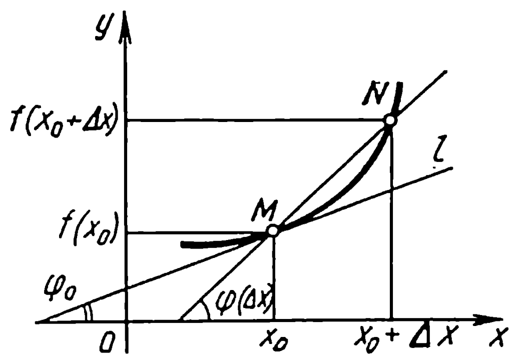


Fig. 1

passes through the point $M(x_0, f(x_0))$ *is a* ***tangent*** *to the graph of the function* $y = f(x)$ *at the point* M.

Theorem 1. *If the function* $y = f(x)$ *has a derivative* $f'(x_0)$ *at the point* x_0, *then the graph of the function has a tangent at the point* $M(x_0, f(x_0))$ *and* $f'(x_0)$ *is the slope of the tangent, i.e., the equation of the tangent line can be written in the form*

$$y - f(x_0) = f'(x_0)(x - x_0).$$

If the function $y = f(x)$ is continuous at x_0 and

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \infty,$$

then we say that the function has an *infinite derivative* at x_0. In that case the tangent to the graph at the point M_0 is parallel to the y-axis and its equation is $x = x_0$.

5. One-sided derivatives. If there is a limit

$$\lim_{\Delta x \to +0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad \left(\lim_{\Delta x \to -0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right),$$

then it is called a *right-hand* (*left-hand*) *derivative* of the function $y = f(x)$ at the point x_0 and is designated as $f'(x_0 + 0)$ $[f'(x_0 - 0)]$ respectively.

If there are $f'(x_0 + 0)$ and $f'(x_0 - 0)$ and they are equal, then there is $f'(x_0)$ and it is equal to $f'(x_0 + 0)$. Conversely, if there is $f'(x_0)$, then there are $f'(x_0 + 0)$

and $f'(x_0 -)$, and in that case $f'(x_0+0)=f'(x_0-0)=f'(x_0)$.

6. Rules of differentiation

Theorem 2. *If $u(x)$ and $v(x)$ have derivatives at the point x_0, then the sum, the difference, the product and the quotient of these functions (the quotient under the condition that $v(x_0) \neq 0$) also have derivatives at the point x_0, and the following equalities hold true at the point x_0:*

$$(u+v)' = u'+v', \quad (u-v)' = u'-v',$$

$$(uv)' = u'v+uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}.$$

7. The derivative of the inverse function

Theorem 3. *If the function $y=f(x)$ is strictly monotonic and continuous in a neighbourhood of the point x_0, has a derivative at the point x_0 and $f'(x_0) \neq 0$, then there is an inverse function $x=f^{-1}(y)$ which is defined in a neighbourhood of the point $y_0 = f(x_0)$ and has a derivative at the point y_0; in that case*

$$(f^{-1}(y_0))' = 1/f'(x_0). \tag{1}$$

The physical interpretation of formula (1) is as follows: the derivative $(f^{-1}(y_0))'$ is the rate of variation of the variable x relative to the variation of the variable y, and $f'(x_0)$ is the rate of variation of the variable y relative to the variation of the variable x. It is clear that these quantities are mutually inverse.

8. The derivative of a composite function

Theorem 4. *If the function $t=\varphi(x)$ has a derivative $\varphi'(x_0)$ at the point x_0 and the function $y=\psi(t)$ has a derivative $\psi'(t_0)$ at the point $t_0=\varphi(x_0)$, then the composite function $y=\psi(\varphi(x)) \equiv f(x)$ has a derivative at x_0, and*

$$f'(x_0) = \psi'(\varphi(x_0))\,\varphi'(x_0). \tag{2}$$

The physical interpretation of formula (2) is the following: the derivative $\varphi'(x_0)$ is the rate of variation of the variable t relative to the variation of the variable x and the derivative $\psi'(t_0)$ is the rate of variation of the variable y relative to the variation of the variable t. It is clear that the rate $f'(x_0)$ of variation of the variable y relative to the variation of the variable x is equal to the product of the rates $\psi'(t_0)$ and $\varphi'(x_0)$. (If t moves k times as fast as x and y moves l times as fast as t, then y moves kl times as fast as x.)

9. The derivative of a function defined parametrically. Assume that the functions

$$x = \varphi(t) \text{ and } y = \psi(t) \tag{3}$$

are defined on an interval of variation of the variable t which we shall consider to be a *parameter*. Let the function $x = \varphi(t)$ be strictly monotonic on that interval. Then there is an inverse function $t = \varphi^{-1}(x)$ and, substituting this inverse function into the equation $y = \psi(t)$, we obtain

$$y = \psi(\varphi^{-1}(x)) \equiv f(x).$$

Thus the variable y is a composite function of the variable x. The representation of the function $y = f(x)$ by means of equations (3) is *parametric*.

We can interpret equations (3) as the relationship between the coordinates of a point moving on the xy-plane and time t. Interpreted in this way, the graph of the function $y = f(x)$ is the path of the point.

If the functions $x = \varphi(t)$ and $y = \psi(t)$ have derivatives $\varphi'(t) \neq 0$ and $\psi'(t)$, then the function $y = f(x)$ also has a derivative, and

$$f'(x) = \left.\frac{\psi'(t)}{\varphi'(t)}\right|_{t=\varphi^{-1}(x)}. \tag{4}$$

Note that the existence of the derivative $\varphi'(t)$ of a definite sign is a sufficient condition for the strict monotonicity of the function $x = \varphi(t)$ and, consequently, for the existence of the function $y = f(x)$ represented parametrically.

10. The derivative of a vector function. If every value of the variable $t \in T$ (T is a number set) is associated with a vector $\mathbf{r}$, then we say that a *vector function* $\mathbf{r} = \mathbf{r}(t)$ is defined on the set T.

Definition. *The vector* $\mathbf{a}$ *is the* ***limit of the vector function*** $\mathbf{r} = \mathbf{r}(t)$ *at the point* t_0 *if* $\lim\limits_{t \to t_0} |\mathbf{r}(t) - \mathbf{a}| = 0$.

Definition. *The* ***derivative of the vector function*** $\mathbf{r} = \mathbf{r}(t)$ *at the point* t *is the limit* $\lim\limits_{\Delta t \to 0} \frac{1}{\Delta t}(\mathbf{r}(t + \Delta t) - \mathbf{r}(t))$ *(provided that it exists).*

The derivative of the vector function $\mathbf{r}(t)$ is designated as $\mathbf{r}'(t)$.

11. Physical meaning of a vector function and its derivative. We can define the position of the point M in

space by its three coordinates or by the vector $\mathbf{r} = \overrightarrow{OM}$, whose initial point coincides with the origin of coordinates and the terminal point with the point M (Fig. 2). If the point M moves, then the vector $\mathbf{r}$ varies with time t. We can thus describe the movement of the point by the vector function $\mathbf{r} = \mathbf{r}\,(t)$, where t varies on some interval $[a, b]$. The set of endpoints of the vectors $\mathbf{r}\,(t)$ (where $t \in [a, b]$) is the path of movement of the point [it is also known as the *hodograph* of the vector function $\mathbf{r} = \mathbf{r}\,(t)$].

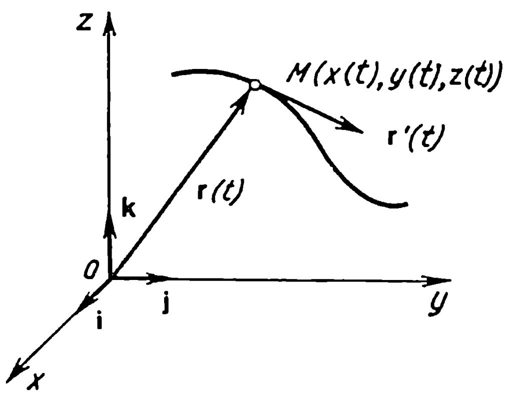


Fig. 2

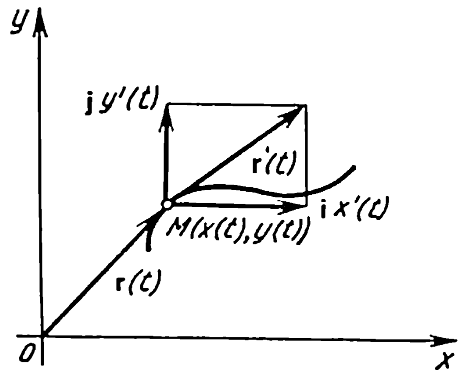


Fig. 3

The derivative $\mathbf{r}'\,(t)$ is the vector of the instantaneous velocity of the point at the moment t. The vector $\mathbf{r}'\,(t)$ is directed along the tangent to the path.

If we designate the coordinates of the point M at the moment t as $x\,(t)$, $y\,(t)$, $z\,(t)$ and the unit vectors of the coordinate axes as $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ then we can represent the vector function $\mathbf{r} = \mathbf{r}\,(t)$ in the form

$$\mathbf{r} = \mathbf{i}x\,(t) + \mathbf{j}y\,(t) + \mathbf{k}z\,(t),$$

and the derivative $\mathbf{r}'\,(t)$ in the form

$$\mathbf{r}'\,(t) = \mathbf{i}x'\,(t) + \mathbf{j}y'\,(t) + \mathbf{k}z'\,(t).$$

Similarly, the movement of the point M on the xy-plane can be described by the vector function $\mathbf{r} = \mathbf{i}x\,(t) + \mathbf{j}y\,(t)$.

If $x'\,(t)$ is of a definite sign, say, $x'\,(t) > 0$, then the path of the point M is the graph of the function $y = f\,(x)$ defined by the parametric equations $x = x\,(t)$ and $y = y\,(t)$.

The coordinates of the velocity vector $\mathbf{r}'\,(t)$ are $x'\,(t)$ and $y'\,(t)$ and the tangent of the angle between the vector

$\mathbf{r}'(t)$ and the x-axis, i.e. the slope of the tangent to the graph of the function $y = f(x)$, is $y'(t)/x'(t)$ (Fig. 3). Then we have again obtained expression (4) for the derivative of the function defined parametrically.

For $x'^2(t) + y'^2(t) \neq 0$ the vector $\mathbf{n}(t) = \{-y'(t), x'(t)\}$ is the *vector of the normal* to the graph of the function $y = f(x)$ at the point $M(x(t), y(t))$, i.e. the direction vector of the straight line passing through the point M at right angles to the tangent to the graph at that point (this straight line is known as a *normal*).

II. Control Questions and Assignments

1. What is the increment of the function $y = f(x)$ at the point x_0?

2. On what argument does the divided difference $\frac{\Delta y}{\Delta x}$ depend? What is the domain of definition of the function $\frac{\Delta y}{\Delta x}$?

3. Give a definition of the derivative of the function $y = f(x)$ at the point x_0.

4. Using the definition of the derivative, derive formulas for the derivatives of the functions x^n (n is a natural number), $\sin x$, $\cos x$, $\log_a x$, a^x.

5. What is the physical meaning of the derivative of the function $y = f(x)$ at the point x_0? What movement of a point is described by the equation $y = v_0 x + y_0$ (x is time and v_0 and y_0 are constants)?

6. What is the geometrical meaning of the derivative of the function $y = f(x)$ at the point x_0? Give a definition of the tangent to the graph of the function $y = f(x)$ at the point $(x_0, f(x_0))$ and write the equation of the tangent.

7. When do we say that a function has an infinite derivative at the point x_0? Give an example of a function whose graph has a vertical tangent at some point.

8. What are the one-sided derivatives of a function at a point? What is the relation between the one-sided derivatives and the derivative of a function at a point? Give an example of a function which has one-sided derivatives at a point but has no derivative at that point.

9. Derive formulas for the derivatives of the sum, the difference, the product and the quotient of two functions. Using them, derive formulas for the derivatives of the functions $\tan x$, $\cot x$, $\sinh x$, $\cosh x$, $\tanh x$ and $\coth x$.

10. Formulate the theorem on the derivative of an inverse function. What can you say of the derivative of an inverse function if all the conditions of the theorem are satisfied, except for the condition $f'(x_0) \neq 0$ (i.e. the condition $f'(x_0) = 0$ is satisfied)? Give an example of such a case. What is the physical interpretation of the formula for the derivative of an inverse function? Using this formula, derive formulas for the derivatives of inverse trigonometric functions.

11. Formulate the theorem on the derivative of a composite function. Is that theorem applicable to the function $y = \sin^2(\sqrt[3]{x^2})$ at the point $x = 0$? Does the derivative of this function exist at the point $x = 0$? What is the physical interpretation of the derivative of a composite function? Using this formula, derive a formula for the derivative of the function x^α (α is any number).

12. What is the parametric definition of a function? Under what conditions does formula (4) for the derivative of a function defined parametrically hold true?

13. What is a vector function? Give definitions of a limit and of the derivative of a vector function. What is the physical meaning of a vector function and its derivative?

14. Using the definition of the derivative of a vector function, derive the formula $\mathbf{r}'(t) = \mathbf{i}x'(t) + \mathbf{j}y'(t) + \mathbf{k}z'(t)$.

15. What are the coordinates of the unit vector of the normal to the hodograph of the vector function $\mathbf{r} = \mathbf{i}x(t) + \mathbf{j}y(t)$ at the point $M(x(t), y(t))$?

III. Worked Problems

1. Using the definition of the derivative, find the derivative of the function $y = x^3$ at the point $x = 1$.

△ We find the increment of the function $y = x^3$ at the point $x = 1$:

$$\Delta y = (1 + \Delta x)^3 - 1 = 3\Delta x + 3(\Delta x)^2 + (\Delta x)^3.$$

From this we obtain $\frac{\Delta y}{\Delta x} = 3 + 3\Delta x + (\Delta x)^2$ and, consequently, $y'(1) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 3$. ▲

2. On the interval $0 \leqslant t \leqslant 1$ compare the instantaneous and average velocities of two points whose rectilinear

motions are defined by the equations $s_1 = t^2$ and $s_2 = 2t^4$ $(t \geqslant 0)$.

△ We find the instantaneous velocities of the points at the moment t: $v_1(t) = s_1'(t) = 2t$, $v_2(t) = s_2'(t) = 8t^3$. From this we obtain $v_1(0) = v_2(0)$, $v_1(1/2) = v_2(1/2)$, $v_1(t) > v_2(t)$ for $0 < t < 1/2$, $v_1(t) < v_2(t)$ for $t > 1/2$. The average velocity of the first point on the time interval $0 \leqslant t \leqslant 1$ is $v_{1\,\text{av}} = \frac{s_2(1) - s_2(0)}{1} = 1$. Similarly, $v_{2\,\text{av}} = \frac{s_1(1) - s_1(0)}{1} = 2$. Thus $v_{1\,\text{av}} < v_{2\,\text{av}}$. ▲

3. Derive an equation of the tangent to the graph of the function $y = \cos x$ at the point with abscissa $x = \pi/6$.

△ We have $x_0 = \pi/6$, $f(x_0) = \cos(\pi/6) = \sqrt{3}/2$, $f'(x_0) = -\sin(\pi/6) = -1/2$. Therefore the required equation of the tangent can be written in the form

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right). \ ▲$$

4. Find the one-sided derivatives of the function $f(x) = |x - x_x|\, g(x)$ at the point x_0, where $g(x)$ is a function continuous at the point x_0. Does the function $f(x)$ possess a derivative at the point x_0?

△For $\Delta x > 0$ the increment of the function at the point x_0 has the form

$$\begin{aligned} \Delta y &= f(x_0 + \Delta x) - f(x_0) \\ &= |x_0 + \Delta x - x_0|\, g(x_0 + \Delta x) - 0 \\ &= g(x_0 + \Delta x)\, \Delta x, \end{aligned}$$

whence $\frac{\Delta y}{\Delta x} = g(x_0 + \Delta x)$. Since $g(x)$ is continuous at the point x_0, we have $\lim\limits_{\Delta x \to +0} \frac{\Delta y}{\Delta x} = g(x_0)$. Thus $f'(x_0 + 0) = g(x_0)$.

Similarly, for $\Delta x < 0$ we get $\Delta y = -g(x_0 + \Delta x)\, \Delta x$, whence

$$\lim_{\Delta x \to -0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to -0} (-g(x_0 + \Delta x)) = -g(x_0),$$

i.e. $f'(x_0 - 0) = -g(x_0)$. If $g(x_0) \neq 0$, then $f'(x_0 + 0) \neq f'(x_0 - 0)$, and, hence, the function $f(x)$ does not have a derivative at the point x_0. Now if $g(x_0) = 0$, then $f'(x_0 + 0) = f'(x_0 - 0) = 0$ and, consequently, the

function $f(x)$ possesses a derivative at the point x_0 and $f'(x_0) = 0$. ▲

5. Calculate the derivative of each of the following functions:

(a) $y = \dfrac{x^2 \sin x}{\ln x}$ $(x > 0,\ x \neq 1)$,

(b) $y = \cos(2^x - x^3)$ $(-\infty < x < \infty)$.

△(a) Using the rules of differentiation of the product and the quotient and the table of derivatives, we get

$$y'(x) = \frac{(x^2 \sin x)' \ln x - x^2 \sin x\, (\ln x)'}{\ln^2 x}$$

$$= \frac{(x^2 \cos x + 2x \sin x) \ln x - x^2 \sin x \dfrac{1}{x}}{\ln^2 x}$$

$$= \frac{x (x \cos x + 2 \sin x) \ln x - x \sin x}{\ln^2 x} \quad (x > 0,\ x \neq 1).$$

(b) We can represent the function $y = \cos(2^x - x^3)$ in the form $y = \cos t$, where $t = 2^x - x^3$. Using the rule of differentiation of a composite function, we get

$$y'(x) = (\cos t)' \big|_{t = 2^x - x^3} (2^x - x^3)'$$

$$= -\sin(2^x - x^3)(2^x \ln 2 - 3x^2) \quad (-\infty < x < \infty). \ ▲$$

6. Find the derivative $y'(x)$ of the function

$$y = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

and find whether $y'(x)$ is continuous at the point $x = 0$.

△For $x \neq 0$ the derivative $y'(x)$ can be found by differentiating the function $x^2 \sin(1/x)$ according to the rule of differentiation of a product. This yields

$$y'(x) = 2x \sin(1/x) - \cos(1/x) \quad (x \neq 0).$$

The expression obtained is not defined for $x = 0$. This does not mean, however, that $y'(0)$ does not exist since the expression for $y'(x)$ was obtained under the condition $x \neq 0$. To find $y'(0)$, we use the definition of the derivative. The increment Δy of the function $y(x)$ at the point $x = 0$ is $(\Delta x)^2 \sin(1/\Delta x)$ and therefore

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \Delta x \sin \frac{1}{\Delta x} = 0, \text{ i.e. } y'(0) = 0.$$

Thus $y'(x)$ exists at all points:

$$y'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

To test $y'(x)$ for continuity at the point $x = 0$ we consider $\lim\limits_{x\to 0} y'(x)$. It is clear that $\lim\limits_{x\to 0} 2x \sin(1/x) = 0$ but $\lim\limits_{x\to 0} \cos(1/x)$ does not exist. Therefore $\lim\limits_{x\to 0} y'(x)$ does not exist either. Thus $y'(x)$ is discontinuous at the point $x = 0$ which is a point of discontinuity of the second kind of the function $y'(x)$. ▲

7. Prove that the equations $x = \cos t$, $y = \sin t$ $(0 \leqslant t \leqslant \pi)$ define parametrically the function $y = f(x)$. Find the derivative $f'(x)$ of this function.

△ The function $x = \cos t$ is strictly monotonic (decreasing) on the interval $0 \leqslant t \leqslant \pi$ and, consequently, has an inverse. Substituting this inverse function into the equation $y = \sin t$, we get a function of the form $y = f(x)$. In this case the inverse function is in explicit form: $t = \arccos x$ and therefore, for $f(x)$, we get an expression $f(x) = \sin(\arccos x)$ $(-1 \leqslant x \leqslant 1)$. We can also write this function in the form $f(x) = \sqrt{1 - x^2}$ $(-1 \leqslant x \leqslant 1)$ (explain why). Calculate the derivative $f'(x)$ by two techniques: (a) using the explicit expression, (b) using the formula for the derivative of a function defined parametrically. We have

(a) $f'(x) = -\dfrac{x}{\sqrt{1-x^2}}$ $(-1 < x < 1)$,

(b) $f'(x) = \dfrac{\cos t}{-\sin t}$ $(t \neq 0,\ t \neq \pi)$.

Since $\cos t = x$, $\sin t = \sqrt{1-\cos^2 t} = \sqrt{1-x^2}$ for $0 \leqslant t \leqslant \pi$, the second expression for $f'(x)$ yields the first: $f'(x) = -\dfrac{x}{\sqrt{1-x_2}}$ $(x \neq \pm 1$, or $-1 < x < 1)$. ▲

8. Prove that if the vector functions $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ have derivatives, then the following formula holds true for the derivative of the scalar product $(\mathbf{r}_1(t)\,\mathbf{r}_2(t))$:

$$(\mathbf{r}_1(t)\,\mathbf{r}_2(t))' = (\mathbf{r}_1'(t)\,\mathbf{r}_2(t)) + (\mathbf{r}_1(t)\,\mathbf{r}_2'(t)).$$

△ Let $\mathbf{r}_1(t) = \mathbf{i}x_1(t) + \mathbf{j}y_1(t) + \mathbf{k}z_1(t)$, $\mathbf{r}_2(t) = \mathbf{i}x_2(t) + \mathbf{j}y_2(t) + \mathbf{k}z_2(t)$. Then $(\mathbf{r}_1(t)\,\mathbf{r}_2(t)) = x_1(t)\,x_2(t) + y_1(t)\,y_2(t) + z_1(t)\,z_2(t)$. We use the fact that if

$\mathbf{r}_i(t)$ $(i=1, 2)$ has a derivative, then $x_i(t)$, $y_i(t)$ and $z_i(t)$ also have derivatives, and in that case $\mathbf{r}_i'(t) = \mathbf{i}x_i'(t) + \mathbf{j}y_i'(t) + \mathbf{k}z_i'(t)$ (see Exercise 20). We obtain

$$\begin{aligned}(\mathbf{r}_1(t)\,\mathbf{r}_2(t))' &= x_1'(t)\,x_2(t) + x_1(t)\,x_2'(t) + y_1'(t)\,y_2(t)\\ &\quad + y_1(t)\,y_2'(t) + z_1'(t)\,z_2(t) + z_1(t)\,z_2'(t)\\ &= \{x_1'(t)\,x_2(t) + y_1'(t)\,y_2(t) + z_1'(t)\,z_2(t)\}\\ &\quad + \{x_1(t)\,x_2'(t) + y_1(t)\,y_2'(t) + z_1(t)\,z_2'(t)\}\\ &= (\mathbf{r}_1'(t)\,\mathbf{r}_2(t)) + (\mathbf{r}_1(t)\,\mathbf{r}_2'(t)). \quad \blacktriangle\end{aligned}$$

IV. Problems and Exercises for Independent Work

1. Write the expression for $\Delta y = f(x_0 + \Delta x) - f(x)$ and find the domain of definition of the function Δy if (a) $f(x) = \arcsin x$, $x_0 = 1/2$, (b) $f(x) = \arccos x$, $x_0 = 0$, (c) $f(x) = \ln x$, $x_0 = 2$, (d) $f(x) = \sin x$, $x_0 = 2\pi$.

2. Using the definition of the derivative, find the derivative of the following function: (a) $y = x$ at the point $x = 1$, (b) $y = x^2$ at the point $x = x_0$, (c) $y = \sqrt{x}$ at the point $x = 4$, (d) $y = x\,|x|$ at the point $x = 0$,

(e) $y = \begin{cases} \dfrac{1-\cos x}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$ at the point $x = 0$.

3. The equations of the rectilinear motion of two points have the form (a) $s_1 = t$, $s_2 = t^2$ $(t \geqslant 0)$, (b) $s_1 = t$, $s_2 = t^3$ $(t \geqslant 0)$, (c) $s_1 = \ln t$, $s_2 = \sqrt{t}$ $(t \geqslant 1)$ (t is time, s_1 and s_2 are the distances traversed by the first and the second point during the time t). Compare the instantaneous velocities of the two points and their average velocities on the time intervals $0 \leqslant t \leqslant 1$ and $1 \leqslant t \leqslant 2$ for cases (a) and (b) and on the time intervals $1 \leqslant t \leqslant 4$ and $1 \leqslant t \leqslant 25$ for case (c).

4. Set up an equation of the tangent to the graph of the function $y = f(x)$ at the point with abscissa x_0 if (a) $f(x) = \sin x$, $x_0 = 0$, (b) $f(x) = x^2$, $x_0 = 1$, (c) $f(x) = \sqrt[3]{x}$, $x_0 = 0$, (d) $f(x) = \arctan x$, $x_0 = 1$.

5. Find the point of intersection of the tangents to the graph of the function $y = f(x)$ at the points with abscissas x_1 and x_2 if (a) $f(x) = \cos x$, $x_1 = \pi/6$, $x_2 = \pi/2$, (b) $f(x) = e^x$, $x_1 = 0$, $x_2 = 1$, (c) $f(x) = \arcsin x$, $x_1 = 0$, $x_2 = 1/2$.

6. Set up equations of the tangents to the graph of the function $y = \sqrt{x}$ which pass through the point $(2, 3/2)$.

7. Find the one-sided derivatives $f'(x_0+0)$ and $f'(x_0-0)$ and compare them if

(a) $f(x)=|x|$, $x_0=0$, (b) $f(x)=|x|$, $x_0=1$,
(c) $f(x)=x^2 \operatorname{sgn} x$, $x_0=0$, (d) $f(x)=\sqrt{\sin^2 x}$, $x_0=0$,
(e) $f(x)=|x|\sin x$, $x_0=0$,
(f) $f(x)=\left|x-\frac{\pi}{2}\right|\cos x$, $x_0=\frac{\pi}{2}$,
(g) $f(x)=|x-1|e^x$, $x_0=1$.

Does the derivative $f'(x_0)$ exist in each case?

8. Find $y'(x)$ if (a) $y=x^2$, (b) $y=\sqrt{x}$, (c) $y=1/x$, (d) $y=2\sqrt[3]{x^2}-3/\sqrt{x}$, (e) $y=\log_2 x^3+\log_3 x^2$ (calculate $y'(1)$), (f) $y=2^x+(1/2)^x$, (g) $y=\sin x-\cos x$ (calculate $y'(0)$ and $y'(\pi/4)$), (h) $y=\tan x-\cot x$, (i) $y=\arcsin x+\arccos x$ (explain the result obtained), (j) $y=\arctan x+\operatorname{arccot} x$ (explain the result obtained).

9. Prove that if $u(x)$ and $v(x)$ have derivatives at the point x and $u(x)>0$, then the function $[u(x)]^{v(x)}$ also has a derivative at the point x, and in that case we have

$$[u(x)^{v(x)}]'=v(x)\,u(x)^{v(x)-1}u'(x)+u(x)^{v(x)}\ln u(x)\,v'(x).$$

10. Find $y'(x)$ if ($a>0$ in every case):

(a) $y=\frac{ax+b}{cx+d}$, $y=\sqrt{x^2-a^2}$, $y=x\sqrt[3]{x^2+1}$,
(b) $y=\sqrt{x+\sqrt{x+\sqrt{x}}}$, $y=\sin^2(\cos x)+\cos^2(\sin x)$,
(c) $y=\sin[\sin(\sin x)]$, $y=\frac{\tan x}{\cot 2x}$, $y=2^{\cos x+\tan x}$,
(d) $y=e^x\sin x$, $y=e^{x^2}\cos 2x$, $y=e^{e^x}+x^{e^x}$,
(e) $y=x^x$, $y=\ln[\ln(\ln x)]$, $y=\frac{1}{2a}\ln\frac{x-a}{x+a}$,
(f) $y=\ln|x|$, $y=\ln\left(x+\sqrt{x^2\pm a^2}\right)$, $y=\ln\sin x$,
(g) $y=\sin(\ln x)$, $y=\arcsin\frac{x}{a}$, $y=\frac{1}{a}\arctan\frac{x}{a}$,
(h) $y=\arctan\frac{1+x}{1-x}$ (compare with the derivative of the function $y=\arctan x$ and explain the result obtained).
(i) $y=\arccos(1/x)$, $y=\arcsin(\sin x)$, $y=\arctan(\tan x)$,
(j) $y=\sin(\arcsin x)$, $y=\cot(\operatorname{arccot} x)$,
(k) $y=\ln\frac{x+a}{\sqrt{x^2+b^2}}+\frac{a}{b}\arctan\frac{x}{b}$, $y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}$,

(l) $y=\frac{\arccos x}{x}+\frac{1}{2}\ln\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}$, $y=\arctan\sqrt{x^2-1}-\frac{\ln x}{\sqrt{x^2-1}}$,

(m) $y=\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{1}{2}\ln\frac{1-x}{1+x}$, $y=\ln(e^x+\sqrt{1+e^{2x}})$,

(n) $y=\arctan(x+\sqrt{1+x^2})$, $y=(\sin x)^{\cos x}$, $y=\sinh(\tan x)$, $y=\tanh(\cos x)$,

(o) $y=\ln(\sinh x)$, $y=\log(\cosh x)$, $y=\arctan(\tanh x)$, $y=\ln\left(\coth\frac{x}{2}\right)$.

11. It is known that $\varphi(x)$, $\psi(x)$ and $f(x)$ have derivatives. Find $y'(x)$ if

(a) $y=\sqrt{\varphi^2(x)+\psi^2(x)}$,

(b) $y=\log_{\varphi(x)}\psi(x)$ $(0<\varphi(x,)\ \varphi(x)\neq 1,\ \psi(x)>0)$,

(c) $y=f(x^2)+f(x^{-2})$, (d) $y=f(f(x))$.

12. Use induction to prove that if $f_1(x), f_2(x), \ldots, f_n(x)$ have derivatives at the point x, then the sum $\sum_{i=1}^{n} f_i(x)$ and the product $f_1(x)f_2(x)\ldots f_n(x)$ also have derivatives at the point x, and in that case we have

$$\left(\sum_{i=1}^{n} f_i(x)\right)'=\sum_{i=1}^{n} f_i'(x),\quad (f_1(x)f_2(x)\ldots f_n(x))'$$
$$=\sum_{i=1}^{n} f_1(x)\ldots f_i'(x)\ldots f_n(x).$$

13. Prove that there is a rule of differentiation of the nth-order determinants:

$$\begin{vmatrix} f_{11}(x) & \ldots & f_{1n}(x) \\ \cdot & \cdots & \cdot \\ f_{k1}(x) & \ldots & f_{kn}(x) \\ \cdot & \cdots & \cdot \\ f_{n1}(x) & \ldots & f_{nn}(x)\end{vmatrix}'=\sum_{k=1}^{n}\begin{vmatrix} f_{11}(x) & \ldots & f_{1n}(x) \\ \cdot & \cdots & \cdot \\ f_{k1}'(x) & \ldots & f_{kn}'(x) \\ \cdot & \cdots & \cdot \\ f_{n1}(x) & \ldots & f_{nn}(x)\end{vmatrix}.$$

14. Can we use the rule of differentiation of the product of two functions $u(x)$ and $v(x)$ at the point x_0 if

(a) $u(x)=x$, $v(x)=|x|$, $x_0=0$,

(b) $u(x)=x$, $v(x)=|x|$, $x_0=1$,

(c) $u(x)=\sin x$, $v(x)=\operatorname{sgn} x$, $x_0=1$,

(d) $u(x) = x^2$, $v(x) = \operatorname{sgn} x$, $x = 0$,

(e) $u(x) = x^3$, $v(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$ $x_0 = 0$?

Does the product $u(x)\,v(x)$ have a derivative at the point x_0 in each case?

15. Find out whether each of the following statements is true:

I. "If $u(x)$ has a derivative at the point x_0 and $v(x)$ does not have a derivative at the point x_0, then (a) $y(x) + v(x)$ does not have a derivative at the point x_0, (b) $u(x)v(x)$ does not have a derivative at the point x_0".

II. "If $u(x)$ and $v(x)$ do not have derivatives at the point x_0, then (a) $u(x) + v(x)$ does not have a derivative at the point x_0, (b) $u(x)\,v(x)$ does not have a derivative at the point x_0".

(If the statement is not true, then give an appropriate example.)

16. Find out whether the following statement is true: "If $f(x) < g(x)$, then $f'(x) < g'(x)$".

17. Derive formulas for the following sums:

$$P_n = 1 + 2x + 3x^2 + \ldots + nx^{n-1},$$
$$Q_n = 1^2 + 2^2x + 3^2x^2 + \ldots + n^2x^{n-1}.$$

18. Depict the path of the point whose motion over the xy-plane is defined by the equations

(a) $x = t$, $y = t^2$, $-\infty < t < \infty$,

(b) $x = \cos^2 t$, $y = \sin^2 t$, $0 \leqslant t < \infty$,

(c) $x = a \cos t$, $y = b \sin t$, $0 \leqslant t < \infty$,

(d) $x = a \cosh t$, $y = b \sinh t$, $-\infty < t < \infty$,

(e) $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $-\infty < t < \infty$,

(f) $x = e^t$, $y = e^{2t}$, $-\infty < t < \infty$.

In each case indicate the interval of variation of the parameter t on which the equations define the function $y = f(x)$ and find the derivative of this function using formula (4). In cases (a), (b), (c), (d) and (f) express $f(x)$ in explicit form and compare the explicit expression for $f'(x)$ with the expression obtained from formula (4). In cases (c) and (d) derive an equation of the tangent and the normal to the curve at the point $t = 0$.

19. Let $\mathbf{r}(t) = \mathbf{i}x(t) + \mathbf{j}y(t) + \mathbf{k}z(t)$ and $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ be constant vectors. Prove the following statement: "For $\lim\limits_{t \to t_0} \mathbf{r}(t) = \mathbf{a}$, it is necessary and sufficient

that $\lim\limits_{t\to t_0} x(t) = a_1$, $\lim\limits_{t\to t_0} y(t) = a_2$, $\lim\limits_{t\to t_0} z(t) = a_3$".

20. Using the result obtained in the preceding problem, prove the following statement: "For the vector function $\mathbf{r}(t) = \mathbf{i}x(t) + \mathbf{j}y(t) + \mathbf{k}z(t)$ to have a derivative $\mathbf{r}'(t)$ at the point t, it is necessary and sufficient that the scalar functions $x(t)$, $y(t)$ and $z(t)$ have derivatives at the point t. In that case $\mathbf{r}'(t) = \mathbf{i}x'(t) + \mathbf{j}y'(t) + \mathbf{k}z'(t)$".

21. Prove that the following rules of differentiation are true for vector functions:

$$(\mathbf{r}_1(t) + \mathbf{r}_2(t))' = \mathbf{r}_1'(t) + \mathbf{r}_2'(t),$$

$$(f(t)\,\mathbf{r}(t))' = f'(t)\,\mathbf{r}(t) + f(t)\,\mathbf{r}'(t),$$

$$[\mathbf{r}_1(t)\,\mathbf{r}_2(t)]' = [\mathbf{r}_1'(t)\,\mathbf{r}_2(t)] + [\mathbf{r}_1(t)\,\mathbf{r}_2'(t)],$$

where $[\mathbf{r}_1(t)\,\mathbf{r}_2(t)]$ is the vector product of the vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$.

22. The motion of a point in space is defined by the equations

(a) $x = t$, $y = t$, $z = t^2$, $t \geqslant 0$,

(b) $x = R\cos t$, $y = R\sin t$, $z = ht$, $t \geqslant 0$, $R > 0$, $h > 0$ (a helical curve),

(c) $x = t$, $y = t^2$, $z = t^3$, $t \geqslant 0$,

(d) $x = \ln t$, $y = t^2/2$, $z = \sqrt{2t}$, $t \geqslant 1$.

Find the modulus and the direction cosines of the velocity vector at the moment (a) $t = 2$, (b) $t = \pi$, (c) $t = 1$, (d) $t = 2.5$.

4.2. The Differential of a Function

I. Fundamental Concepts and Theorems

1. Differentiability of a Function

Definition. *The function $y = f(x)$ is **differentiable** at the point x_0 if its increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ at that point can be represented as*

$$\Delta y = A\Delta x + \alpha\Delta x, \tag{1}$$

where A is a number and α is a function of the argument Δx infinitesimal and continuous at the point $\Delta x = 0$ (i.e. $\lim\limits_{\Delta x\to 0} \alpha(\Delta x) = \alpha(0) = 0$).

Theorem 5. *For the function $y = f(x)$ to be differentiable at the point x_0, it is necessary and sufficient for the derivative $f'(x_0)$ to exist.*

Note that in this case $A = f'(x_0)$.

The *differential* (or the *first differential*) of the function $y = f(x)$ at point x_0 (differentiable at this point) is a function of the argument Δx: $dy = f'(x_0)\,\Delta x$.

For $f'(x_0) \neq 0$ the differential is the principal (linear with respect to Δx) part of the increment of the function at the point x_0.

The *differential of the independent variable* x is the increment of this variable: $dx = \Delta x$. Thus the differenti-

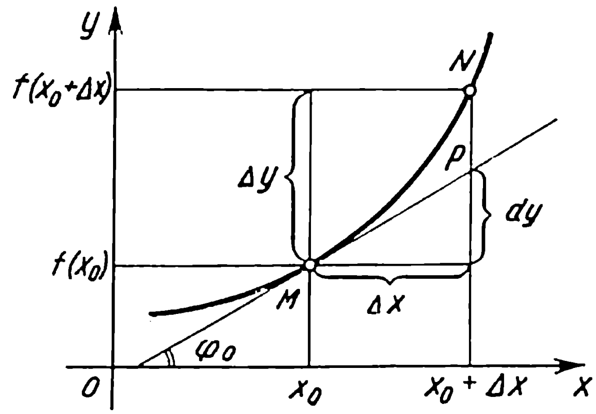


Fig. 4

al of the function $y = f(x)$ at the point x_0 has the form

$$dy = f'(x_0)\,dx, \tag{2}$$

whence we have

$$f'(x_0) = \frac{dy}{dx},$$

i.e. *the derivative of the function $y = f(x)$ at the point x_0 is equal to the ratio of the differential of the function at this point to the differential of the independent variable.*

2. Geometrical and physical meaning of the differential. It is easy to see the geometrical meaning of the differential of a function from Fig. 4 which shows the graph of the function $y = f(x)$ (heavy line) and the tangent MP to the graph at the point $M(x_0, f(x_0))$. The differential dy is equal to the increment of the linear function whose graph is the tangent MP.

If x is time and $y = f(x)$ is the coordinate of the point on the straight line at the moment x, then the differential

$dy = f'(x_0)\,\Delta x$ is equal to the variation of the coordinate which the point would obtain during the time Δx if the velocity of the point on the time interval $[x_0, x_0 + \Delta x]$ were constant and equal to $f'(x_0)$. The variation of the velocity on this interval leads to the fact that, in general, $\Delta y \neq dy$. On small time intervals Δx, however, the variation of velocity is negligible and $\Delta y \cong dy = f'(x_0)\Delta x$.

3. The invariance of the form of the first differential. Assume that the argument x of the function $y = f(x)$ differentiable at the point x_0 is not an independent variable but a function of some independent variable t, i.e. $x = \varphi(t)$, with $x_0 = \varphi(t_0)$ and $\varphi(t)$ is differentiable at the point t_0. Then, as before, the differential of the function $y = f(x)$ has form (2): $dy = f'(x_0)\,dx$, but now dx is not an arbitrary increment of the argument x (as in the case when x is an independent variable) but the differential of the function $x = \varphi(t)$ at the point t_0, i.e. $dx = \varphi'(t_0)\,dt$. This property [formula (2) is retained when $x = \varphi(t)$] is the *invariance of the form of the first differential.*

4. Using the differential for approximations. Since $\Delta y \cong dy$ for small Δx, i.e. $f(x_0 + \Delta x) - f(x_0) \cong f'(x_0)\,\Delta x$, it follows that

$$f(x_0 + \Delta x) \cong f(x_0) + f'(x_0)\,\Delta x. \tag{3}$$

This formula makes it possible to find approximate values of $f(x_0 + \Delta x)$ for small Δx when $f(x_0)$ and $f'(x_0)$ are known. The error caused by the replacement of $f(x_0 + \Delta x)$ by the right-hand side of formula (3) is the smaller the smaller Δx is and, what is more, as $\Delta x \to 0$, this error is infinitesimal of the order of smallness higher than that of Δx.

II. Control Questions and Assignments

1. Give a definition of the differentiability of a function at a given point.
2. Prove the theorem on the relationship between the differentiability of a function at a point and the existence of the derivative at that point.
3. What is the differential of a function at a given point? On what argument does it depend?
4. Can the differential of a function at a point be a constant quantity?

5. For what functions is the differential equal to the increment of the function? Give examples.
6. What is the geometrical meaning of the differential?
7. What is the physical meaning of the differential?
8. What is the invariance of the form of the first differential? Prove that the form of the first differential is invariant.
9. How can the differential of a function be used for approximations?

III. Worked Problems

1. Find the differential of the function $y = x^2 - x + 3$ at the point $x = 2$ in two ways: (a) by isolating the linear, with respect to Δx, part of Δy, (b) by using formula (2).

(a) $\Delta y = f(2 + \Delta x) - f(2) = [(2 + \Delta x)^2 - (2 + \Delta x) + 3] - [2^2 - 2 + 3] = 3\,\Delta x + (\Delta x)^2$. It follows that $dy = 3\,\Delta x$.

(b) $f'(x) = 2x - 1$, $f'(2) = 3$. Consequently, from formula (2) we find that $dy = 3\,dx = 3\,\Delta x$. ▲

2. Find the differential of the function $y = \sin(x^2)$: (a) at the point $x = x_0$, (b) at the point $x = \sqrt{\pi}$, (c) at the point $x = \sqrt{\pi}$ for $dx = -2$.

△(a) According to formula (2) we have $dy\,|_{x=x_0} = f'(x_0)\,dx = \cos(x_0^2)\,2x_0\,dx$.

(b) Setting $x_0 = \sqrt{\pi}$ in the last relation, we get $dy\,|_{x=\sqrt{\pi}} = -2\sqrt{\pi}\,dx$.

(c) We have $dy\Big|_{\substack{x=\sqrt{\pi}\\ dx=-2}} = 4\sqrt{\pi}$. ▲

3. Replacing the increment of the function by its differential, find the approximate value of (a) $\sqrt{0.98}$, (b) $\sin 31°$.

△ (a) Let us consider the function $y(x) = \sqrt{1 + x}$. Since $y(0) = 1$, $y(-0.02) = \sqrt{0.98}$, $y'(x) = \frac{1}{2}(1 + x)^{-1/2}$, $y'(0) = \frac{1}{2}$, we find from formula (3) that $y(-0.02) \cong y(0) + y'(0)(-0.02) = 1 - 0.01 = 0.99$. Thus $\sqrt{0.98} \cong 0.99$.

(b) Consider the function $y = \sin x$. Since $y(30°) = \sin 30° = 1/2$, $y'(30°) = \cos 30° = \sqrt{3}/2$, and $1° =$

$2\pi/360$ (rad) $\cong 0.0175$ (rad), we find from formula (3) that

$$\sin 31^\circ \cong \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{2\pi}{360} \cong 0.5151. \ \blacktriangle$$

IV. Problems and Exercises for Independent Work

23. Express in form (1) the increment of the function (a) $y = e^x$ at the point $x = 0$, (b) $y = \sin x$ at the point $x = \pi/2$, (c) $y = \arctan x$ at the point $x = 0$. Write an expression for the function $\alpha(\Delta x)$.

24. Find the increment and the differential of the function $y = x^3 - x^2 + 1$ at the point $x = 1$ and calculate their values for (a) $\Delta x = 0.01$, (b) $\Delta x = 0.1$, (c) $\Delta x = 1$, (d) $\Delta x = 3$.

25. The rectilinear motion of a point is defined by the equation $s = 2t^2 + t + 1$, where t is in seconds and s in metres. Find the increment and the differential of the path s at the moment $t = 1$ s and compare them for (a) $\Delta t = 0.1$ s, (b) $\Delta t = 0.2$ s, (c) $\Delta t = 1$ s.

26. Find the differential of the function y at the point x if

(a) $y = \sqrt{x}$, (b) $y = 1/x$, (c) $y = \ln(x + \sqrt{x^2+1})$, (d) $y = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|$, (e) $y = \arcsin\frac{x}{a}$, (f) $y = \frac{1}{a}\arctan\frac{x}{a}$, (g) $y = xe^{2x}$, (h) $y = x\sin x + \cos x$.

27. Find $dy|_{x=0}$ and $dy|_{x=1}$ if

(a) $y = \frac{x^3}{3} - \frac{x^2}{2} + x$, (b) $y = \ln(1+x)$, (c) $y = e^x$, (d) $y = \sin(\pi x/2)$, (e) $y = \cos(\pi x/2)$.

28. Construct the graph of the function $y = \ln(1+x)$ and represent dy on the graph for (a) $x = 0$, $dx = 1$, (b) $x = 1$, $dx = 1$, (c) $x = 1$, $dx = 2$.

29. Let $y = \sin x$, where $x = \cos t$. Which of the following equalities hold true: (a) $dy|_{t=\pi/2} = 0$, (b) $dy|_{t=\pi/2} = dx$, (c) $dy|_{t=\pi/2} = -dt$?

30. Using formula (3) and choosing an appropriate value of x_0, find the approximate values of (a) $\cos 151^\circ$, (b) $\arcsin 0.49$, (c) $\log 11$, (d) $\sqrt[3]{1.01}$, (e) $\arctan 1.1$, (e) $e^{0.2}$.

31. Prove the approximate formula (for small x)

$$\sqrt[n]{a^n + n} \cong a + \frac{x}{na^{n-1}} \quad (a > 0).$$

Using this formula, find the approximate values of (a) $\sqrt[3]{9}$, (b) $\sqrt[4]{255}$, (c) $\sqrt[7]{130}$.

4.3. Higher-Order Derivatives and Differentials

I. Fundamental Concepts and Theorems

1. Definition of higher-order derivatives. If the derivative $f'(x)$ of the function $y = f(x)$ is defined in a neighbourhood of the point x_0 and has a derivative at this point, then the derivative of $f'(x)$ is the *second derivative* (or a *second-order derivative*) of the function $y = f(x)$ at the point x_0 and is denoted by one of the following symbols: $f''(x_0)$, $f^{(2)}(x_0)$, $y''(x_0)$, $y^{(2)}(x_0)$.

The third derivative is defined as a derivative of the second derivative and so on. If the concept of the $(n-1)$th derivative has been introduced and if the $(n-1)$th derivative has a derivative at the point x_0, then the derivative is the nth *derivative* (or the nth-*order derivative*) of the function $y = f(x)$ at the point x_0 and is designated as $f^{(n)}(x_0)$ or $y^{(n)}(x_0)$.

Thus higher-order derivatives can be found by mathematical induction from the formula

$$y^{(n)}(x) = [y^{(n-1)}(x)]'.$$

A function which has an nth derivative at the point x_0 is said to be *n times differentiable* at that point. A function which has derivatives of all orders at x_0 is said to be *infinitely differentiable* at that point.

The higher-order derivatives of the vector function $\mathbf{r} = \mathbf{r}(t)$ can also be introduced by induction:

$$\mathbf{r}^{(n)}(t) = [\mathbf{r}^{(n-1)}(t)]'.$$

If $\mathbf{r}(t) = \mathbf{i}x(t) + \mathbf{j}y(t) + \mathbf{k}z(t)$, then $\mathbf{r}^{(n)}(t) = \mathbf{i}x^{(n)}(t) + \mathbf{j}y^{(n)}(t) + \mathbf{k}z^{(n)}(t)$.

If the function $\mathbf{r} = \mathbf{r}(t)$ describes the motion of a point (t is time), then the second derivative $\mathbf{r}''(t)$ is the acceleration at the moment t.

2. The main rules of calculating the nth derivatives.

1. $(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}$.

2. **Leibniz's formula:**

$$(uv)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} u^{(i)} v^{(n-i)}, \quad \text{where } u^{(0)} = u, \; v^{(0)} = v,$$

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, \quad 0! = 1.$$

3. Formulas for the nth derivatives of some functions:

1. $(x^\alpha)^{(n)} = \alpha(\alpha - 1) \dots (\alpha - n + 1)\ x^{\alpha - n}$ ($x > 0$, α is any number).

2. $(a^x)^{(n)} = a^x (\ln a)^n$ ($0 < a \neq 1$), in particular, $(e^x)^{(n)} = e^x$.

3. $(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right)$.

4. $(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right)$.

4. Differentials of higher orders. Assume that x is an independent variable and the function $y = f(x)$ is differentiable in a neighbourhood of the point x_0. The first differential $dy = f'(x)\,dx$ is a function of two variables, x and dx. The *second differential* d^2y of the function $y = f(x)$ at the point x_0 is defined as the differential of the function $dy = f'(x)\,dx$ at the point x_0 under the following conditions: (1^0) dy is regarded as a function of only the independent variable x (in other words, when we calculate the differential of $f'(x)\,dx$ we must calculate the differential of $f'(x)$ considering dx to be a constant factor), (2^0) when the differential of $f'(x)$ is calculated, the increment of the independent variable x is considered to be equal to the initial increment of the argument, i.e. to the same value dx which appears as a factor in the expression $dy = f'(x)\,dx$.

Using this definition, we obtain

$$d^2y\,|_{x=x_0} = d(dy)\,|_{x=x_0} = d[f'(x)]\,|_{x=x_0}\,dx$$
$$= \{[f'(x)]'\,|_{x=x_0}\,dx\}dx = f''(x_0\,(dx)^2,$$

or (writing (dx^2) as dx^2),

$$d^2y\,|_{x=x_0} = f''(x_0)\,dx^2.$$

The differential of the arbitrary nth order of the function $y = f(x)$ is found by mathematical induction from the formula

$$d^ny = d(d^{n-1}y)$$

under the same two conditions as the differential of the second order. In that case, the formula

$$d^ny\,|_{x=x_0} = f^{(n)}(x_0)\,dx^n \quad (dx^n = (dx)^n) \tag{1}$$

holds true, whence we find that

$$f^{(n)}(x_0) = \frac{d^ny}{dx^n}. \tag{2}$$

If x is not an independent variable but a function of some variable t, then formulas (1) and (2) become untrue (noninvariance of the form of higher-order differentials). In particular, for $n = 2$ we have

$$\mathrm{d}^2 y = f''(x)\,\mathrm{d}x^2 + f'(x)\,\mathrm{d}^2 x.$$

II. Control Questions and Assignments

1. Give a definition of the second derivative of the function $y = f(x)$ at the point x_0.
2. Can the second derivative $f''(x_0)$ exist if the first derivative $f'(x_0)$ does not exist?
3. Give an example of a function for which $f'(x_0)$ exists but $f''(x_0)$ does not exist.
4. Give a definition of the nth derivative of the function $y = f(x)$ at the point x_0.
5. The nth derivative of the function at the point x_0 is known to exist. What can we say of the existence of lower-order derivatives at the point x_0 and in the neighbourhood of this point?
6. Give a definition of the nth derivative of a vector function. What is the physical meaning of the second derivative of a vector function which describes the motion of a point?
7. Use mathematical induction to prove the rule of finding the nth derivative of the sum and the difference of two functions.
8. Derive Leibniz's formula.
9. Derive formulas for the nth derivatives of the functions x^α, a^x, $\sin x$, $\cos x$, $\ln x$.
10. Prove that if $f(x)$ is n times differentiable, then

$$\frac{\mathrm{d}^n f(ax+b)}{\mathrm{d}x^n} = a^n f^{(n)}(t)\,\big|_{t=ax+b}.$$

11. Calculate the derivatives of the nth order $(e^{5x})^{(n)}$, $(\sin(3x+2))^{(n)}$, $(\sqrt{1-x})^{(n)}$.
12. Give a definition of the nth-order differential of function $y = f(x)$ at the point x_0.
13. Prove the validity of formula (1) for the nth-order differential in the case when x is an independent variable.
14. Does formula (1) hold true if x is a function of a variable t? For that case derive formulas for $\mathrm{d}^2 y$ and $\mathrm{d}^3 y$.

Prove that formula (1) remains true if x is a linear function of the independent variable t, i.e. $x = at + b$ (a and b are numbers).

III. Worked Problems

1. Find $y^{(10)}$ if $y = x^2e^{3x}$.

△ This function is a product of two functions x^2 and e^{3x}. Using Leibniz's formula, we obtain

$$(x^2e^{3x})^{(10)} = x^2(e^{3x})^{(10)} + \binom{10}{1}(x^2)'(e^{3x})^{(9)}$$
$$+\binom{10}{2}(x^2)^{(2)}(e^{3x})^{(8)} + \ldots + (x^2)^{(10)}e^{3x}.$$

Since $(x^2)^{(n)} = 0$ for $n \geqslant 3$, $(e^{3x})^{(k)} = e^{3x}3^k$, it follows that

$$(x^2e^{3x})^{10} = x^2e^{3x}3^{10} + 10\cdot 2xe^{3x}3^9 + 45\cdot 2e^{3x}3^8$$
$$= 3^9e^{3x}(3x^2 + 20x + 30). \quad \blacktriangle$$

The example considered shows that it is most convenient to use Leibniz's formula in the cases when one of the factors is a polynomial of a not high degree of p. In that case, all terms of Leibniz's formula, beginning with $(p + 2)$, are zero.

2. Find the nth derivative of the function $y = \frac{x^2+1}{x^2-1}$.

△ We can represent this function as $y = 1 + \frac{2}{x^2-1}$. Therefore $y^{(n)} = 1^{(n)} + \left(\frac{2}{x^2-1}\right)^{(n)} = \left(\frac{2}{x^2-1}\right)^{(n)}$. In its turn, $\frac{2}{x^2-1}$ can be decomposed into partial fractions:

$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}.$$

Consequently,

$$y^{(n)} = \left(\frac{1}{x-1}\right)^{(n)} - \left(\frac{1}{x+1}\right)^{(n)}.$$

Let us successively calculate the first, the second and the third derivative of the function $\frac{1}{x-1}$:

$$\left(\frac{1}{x-1}\right)' = [(x-1)^{-1}]' = -1\cdot(x-1)^{-2},$$

$$\left(\frac{1}{x-1}\right)^{2} = (-1)(-2)(x-1)^{-3} = (-1)^2\cdot 2!\,(x-1)^{-3},$$
$$\left(\frac{1}{x-1}\right)^{3} = (-1)^2\cdot 2!\,(-3)(x-1)^{-4} = (-1)^3\cdot 3!\,(x-1)^{-4}.$$

Then it is easy to prove by mathematical induction that

$$\left(\frac{1}{x-1}\right)^{(n)} = (-1)^n\, n!\,(x-1)^{-(n+1)} = \frac{(-1)^n\, n!}{(x-1)^{n+1}}.$$

Similarly

$$\left(\frac{1}{x+1}\right)^{(n)} = \frac{(-1)^n\, n!}{(x+1)^{n+1}}.$$

Thus we have

$$y^{(n)} = (-1)^n\, n!\left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right]. \quad \blacktriangle$$

3. The function $y = f(x)$ is defined parametrically by the equations $x = a\cos t$, $y = a\sin t$, $0 < t < \pi$. Find $f''(x)$.

$\triangle$ We derive a formula for the second derivative of the function $y = f(x)$ defined parametrically by the equations $x = \varphi(t)$, $y = \psi(t)$, considering the functions $\varphi(t)$ and $\psi(t)$ to be twice differentiable and $\varphi(t) \neq 0$.

Since the form of the first differential is invariant, $\mathrm{d}f'(x) = f''(x)\,\mathrm{d}x$, whence $f''(x) = \dfrac{\mathrm{d}f'(x)}{\mathrm{d}x}$. Since $f'(x) = \dfrac{\psi'(t)}{\varphi'(t)}$, it follows that $\mathrm{d}f'(x) = \left(\dfrac{\psi'(t)}{\varphi'(t)}\right)'\mathrm{d}t$. Taking into account that $\mathrm{d}x = \varphi'(t)\,\mathrm{d}t$, we obtain

$$f''(x) = \left(\frac{\psi'(t)}{\varphi'(t)}\right)'\frac{1}{\varphi'(t)}\bigg|_{t=\varphi^{-1}(x)}$$
$$= \frac{\psi''(t)\,\varphi'(t) - \varphi''(t)\,\psi'(t)}{\varphi'^{3}(t)}\bigg|_{t=\varphi^{-1}(x)}. \qquad (3)$$

Setting $\psi = a\sin t$, $\varphi = a\cos t$, $\varphi^{-1}(x) = \arccos(x/a)$ in this formula, we get

$$f''(x) = -\frac{1}{a\sin^3 t}\bigg|_{t=\arccos(x/a)} = -\frac{1}{a\left(1-\dfrac{x^2}{a^2}\right)^{3/2}}$$
$$= -\frac{a^2}{(a^2-x^2)^{3/2}}.$$

In this example we can find an explicit expression for $f(x)$: $f(x) = \sqrt{a^2 - x^2}$ $(-a < x < a)$. It stands to rea-

son that calculating $f''(x)$, we get an expression the same as that obtained using formula (3). ▲

4. The motion of a point in space is defined by the equations $x = R\cos t$, $y = R\sin t$, $z = ht^2/2$, $t \geqslant 0$. Find the moduli of the vectors of the velocity and the acceleration at the moment $t = 1$.

△ By means of the vector function, we can define the motion by the equation

$$\mathbf{r} = \mathbf{i}R\cos t + \mathbf{j}R\sin t + \mathbf{k}ht^2/2, \quad t \geqslant 0.$$

By means of differentiation we find that

$$\mathbf{r}'(t) = -\mathbf{i}R\sin t + \mathbf{j}R\cos t + \mathbf{k}ht \text{ (velocity)},$$
$$\mathbf{r}''(t) = -\mathbf{i}R\cos t - \mathbf{j}R\sin t + \mathbf{k}h \text{ (acceleration)}.$$

Hence we get $|\mathbf{r}'(t)| = \sqrt{R^2 + h^2t^2}$, $|\mathbf{r}'(1)| = \sqrt{R^2 + h^2}$, $|\mathbf{r}''(t)| = \sqrt{R^2 + h^2}$ (the absolute value of the acceleration is constant).

5. Find the second differential of the function $y = \cos 2x$ if (a) x is an independent variable, (b) $x = \varphi(t)$, where $\varphi(t)$ is a twice differentiable function of the independent variable t.

△ (a) $d^2y = y''(x)(dx^2) = -4\cos 2x\,(dx)^2$,

(b) $d^2y = y''(x)(dx)^2 + y'(x)\,d^2x$
$= -4\cos(2\varphi(t))(\varphi'(t)\,dt)^2 - 2\sin(2\varphi(t))\,\varphi''(t)\,(dt)^2 =$
$= -2\,[2\cos(2\varphi(t))\,\varphi'^2(t) + \sin(2\varphi(t))\,\varphi''(t)]\,(dt)^2$. ▲

IV. Problems and Exercises for Independent Work

32. Find the derivatives of the indicated order:
(a) $(e^{-x^2})^{(3)}$, (b) $(\sin ax)^{(10)}$, (c) $(e^{kx})^{(4)}$,
(d) $(f(x^2))^{(3)}$, (e) $(f(e^x))^{(2)}$, (f) $(f(\varphi(x)))^{(3)}$,
(g) $(\sqrt{x})^{(10)}$, (h) $\left(\frac{x^2}{x-1}\right)^{(6)}$, (i) $(x^2\sin 2x)^{(20)}$,
(j) $(x^3\cos 5x)^{(15)}$,
(k) $\left(\frac{x-1}{x+1}\right)^{(8)}$,
(l) $\left(\frac{x}{x^2-1}\right)^{(30)}$,
(m) $(xe^{5x})^{(11)}$, (n) $(\ln 3x)^{(10)}$.

33. Find $y^{(n)}$ if

(a) $y=\sqrt{ax+b}$, (b) $y=\frac{ax+b}{cx+d}$, (c) $y=\sin^2 x$,

(d) $y=\cos^2 x$, (e) $y=\sin^3 x$, (f) $y=\cos^3 x$,

(g) $y=\sin \alpha x \sin \beta x$, (h) $y=\cos \alpha x \cos \beta x$,

(i) $y=x \sin ax$, (j) $y=x^2 \cos ax$,

(k) $y=(ax^2+bx+c)\,e^{kx}$,

(l) $y=\ln \frac{ax+b}{ax-b}$, (m) $y=x \sinh x$,

(n) $y=x^2 \cosh x$,

(o) $y=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n$ (a_i are numbers).

34. Find $f''(x)$ and $f'''(x)$ for the function from Exercise 18 defined parametrically.

35. Express the derivatives of the inverse function $x=f^{-1}(y)$ up to the third order inclusive in terms of the derivatives of the function $y=f(x)$.

36. The motion of a point in space is defined by the equations from Exercise 22. Find the modulus and the direction cosines of the vector of acceleration at the indicated moments.

37. Find the differentials of the indicated order if x is an independent variable: (a) $d^3(x^3)$, (b) $d^4(\sqrt{x-1})$, (c) $d^5(x \ln x)$, (d) $d^{10}(x \sin x)$.

38. Find $d^n y$ if (a) $y=\sinh x$, (b) $y=\cosh(ax)$, (c) $y=x^2 \ln x$.

39. In each of the following cases verify whether the function $y(x)$ satisfies the corresponding equation (C_i are arbitrary numbers):

(a) $y=C_1 \sin kx+C_2 \cos kx$, $y''=k^2y=0$,

(b) $y=C_1e^{kx}+C_2e^{-kx}$, $y''-k^2y=0$,

(c) $y=e^{-\alpha x}(C_1 \cos \beta x+C_2 \sin \beta x)$, $y''+2\alpha y'+(\alpha^2+\beta^2)\,y=0$,

(d) $y=C_1 \sin x+C_2 \cos x+C_3e^x+C_4e^{-x}$, $y^{(4)}-y=0$.

40. Find $f^{(n)}(x_0)$ if $f(x)=(x-x_0)^n\varphi(x)$, where $\varphi(x)$ has a continuous derivative of the order $n-1$ at the point x_0.

41. Prove that the function $f(x)=\begin{cases} e^{-1/x^2} & \text{for } x\neq 0, \\ 0 & \text{for } x=0 \end{cases}$ is infinitely differentiable at the point $x^0=0$.

Chapter 5

The Indefinite Integral

5.1. The Antiderivative and the Indefinite Integral

I. Fundamental Concepts and Theorems

1. Definition of the antiderivative and of the indefinite integral

Definition. *The function $F(x)$ is an* ***antiderivative*** *of the function $f(x)$ on the interval X if $F'(x) = f(x)\ \forall x \in X$.*

Theorem 1. *If $F_1(x)$ and $F_2(x)$ are two arbitrary antiderivatives of $f(x)$ on X, then $F_1(x) - F_2(x) = \text{const}$.*

Corollary. *If $F(x)$ is one of the antiderivaties of $f(x)$ on X, then any other antiderivative $\Phi(x)$ of the function $f(x)$ on X has the form $\Phi(x) = F(x) + C$, where C is a constant.*

Definition. *The set of all antiderivatives of the function $f(x)$ on X is an* ***indefinite integral*** *of the function $f(x)$ on the interval X and is designated as $\int f(x)\,dx$.*

By virtue of the corollary of Theorem 1, $\int f(x)\,dx = F(x) + C$, where $F(x)$ is one of the antiderivatives of $f(x)$, and C is an arbitrary constant.

(Sometimes the symbol $\int f(x)\,dx$ denotes not the whole set of antiderivatives but any one of them.)

2. Principal properties of the indefinite integral

1°. $d \int f(x)\,dx = f(x)\,dx$.

2°. $\int dF(x) = F(x) + C$.

3°. **Linearity of an integral.** If there are antiderivatives of the functions $f(x)$ and $g(x)$ and α and β are any real numbers, then there is an antiderivative of the function $\alpha f(x) + \beta g(x)$, and

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx.$$

II. Control Questions and Assignments

1. Give a definition of an antiderivative of the function $f(x)$ on the interval X.
2. Give examples of functions which have antiderivatives.
3. Give examples of two different antiderivatives of the function $f(x)$.
4. Does every function have an antiderivative? Consider the following example:

$$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ -2 & \text{for } x \leqslant 0. \end{cases}$$

5. Find an antiderivative of the function $f(x) = \sin x$ which assumes a value 10 at the point $x = \pi/2$.
6. It is known that at the point $x = 1$ two antiderivatives of the function $f(x) = e^x$ differ by 2. How much do these antiderivatives differ at the point $x = 100$?
7. The graph of what antiderivative of the function $f(x) = 1/(1 + x^2)$ passes through the point with coordinates $(1, 2\pi)$?

III. Worked Problems

1. Prove that the function

$$\operatorname{sgn} x = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0 \end{cases}$$

has an antiderivative on any interval which does not contain the point $x = 0$ and does not possess an antiderivative on any interval which contains the point $x = 0$.

△On any interval which does not contain the point $x = 0$ the function $\operatorname{sgn} x$ is constant. For instance, on the closed interval $[1, 2]$ the function $\operatorname{sgn} x = 1$ and any antiderivative of the function $\operatorname{sgn} x$ on this interval has the form $F(x) = x + C$, where C is a constant.

Let us consider a closed interval which contains the point $x = 0$, say, $[-1, 3]$. On the half-open interval $[-1, 0)$ any antiderivative of the function $\operatorname{sgn} x$ has the form $-x + C_1$ and on the half-closed interval $(0, 3]$ the antiderivative of $\operatorname{sgn} x$ is $x + C_2$. For any choice of the arbitrary constan ts C_1 and C_2 we get on the interval

$[-1, 3]$ a function which does not have an antiderivative at the point $x = 0$. If we choose $C_1 = C_2$, we get a continuous function $y = |x| + C$ $(C = C_1 = C_2)$ which is not differentiable at the point $x = 0$ either. Thus the function sgn x does not have an antiderivative on the interval $[-1, 3]$.

This example shows that the question concerning the existence of an antiderivative of a function is essentially connected with the interval on which the function is considered. ▲

2. The problem of finding the law of the rectilinear motion of a particle using its velocity furnishes an important example of an antiderivative. The instantaneous velocity $v(t)$ is a derivative of the function $s(t)$ which defines the law of motion of a particle. Therefore, to find the function $s(t)$ using the given velocity $v(t)$, we must find an antiderivative of the function $v(t)$. Any antiderivative of $v(t)$ has the form

$$s(t) = \int v(t)\,\mathrm{d}t + C.$$

Some additional conditions must be taken into account to find the constant C. Assume, for instance, that $v(t) = a(t - t_0) + v_0$ (motion with the acceleration $a = \text{const}$), $s(t_0) = s_0$. Then $s(t) = \frac{a(t-t_0)^2}{2} + v_0(t - t_0) + C$. From the additional condition $s(t_0) = s_0$ we find $C = s_0$, and therefore

$$s(t) = \frac{a(t-t_0)^2}{2} + v_0(t - t_0) + s_0.$$

5.2. Basic Indefinite Integrals

I. Table of Fundamental Indefinite Integrals

I. $\int 0\,\mathrm{d}x = C.$

II. $\int 1\,\mathrm{d}x = x + C.$

III. $\int x^\alpha\,\mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C \quad (\alpha \neq -1).$

IV. $\int \frac{\mathrm{d}x}{x} = \ln|x| + C \quad (x \neq 0).$

V. $\int a^x\,\mathrm{d}x=\frac{a^x}{\ln a}+C \quad (0<a\neq 1), \qquad \int e^x\,\mathrm{d}x=e^x+C.$

VI. $\int \sin x\,\mathrm{d}x=-\cos x+C,$

VII. $\int \cos x\,\mathrm{d}x=\sin x+C.$

VIII. $\int \frac{\mathrm{d}x}{\cos^2 x}=\tan x+C \quad \left(x\neq\frac{\pi}{2}+\pi n,\ n\in\mathbf{Z}\right).$

IX. $\int \frac{\mathrm{d}x}{\sin^2 x}=-\cot x+C \quad (x\neq \pi n,\ n\in\mathbf{Z}).$

X. $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}=\begin{cases}\arcsin x+C,\\ -\arccos x+C\end{cases} \quad (-1<x<1).$

XI. $\int \frac{\mathrm{d}x}{1+x^2}=\begin{cases}\arctan x+C,\\ -\operatorname{arccot} x+C.\end{cases}$

XII. $\int \frac{\mathrm{d}x}{\sqrt{x^2\pm 1}}=\ln\left|x+\sqrt{x^2\pm 1}\right|+C.$

XIII. $\int \frac{\mathrm{d}x}{1-x^2}=\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+C.$

XIV. $\int \sinh x\,\mathrm{d}x=\cosh x+C.$

XV. $\int \cosh x\,\mathrm{d}x=\sinh x+C.$

XVI. $\int \frac{\mathrm{d}x}{\cosh^2 x}=\tanh x+C.$

XVII. $\int \frac{\mathrm{d}x}{\sinh^2 x}=-\coth x+C.$

II. Worked Problems

The following integrals can be reduced to the tabular ones by means of an identity transformation of the element of integration:

1. $\int \frac{x+1}{\sqrt{x}}\,\mathrm{d}x=\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\mathrm{d}x=\frac{2x^{3/2}}{3}+2x^{1/2}+C.$

2. $\int (x^4+1)\,x^3\,\mathrm{d}x=\frac{1}{4}\int (x^4+1)\,\mathrm{d}(x^4+1)$

$$=\frac{1}{8}\int \mathrm{d}(x^4+1)^2=\frac{1}{8}(x^4+1)^2+C.$$

3. $\displaystyle\int \frac{x^2\,dx}{1-x^2} = \int \frac{(x^2-1)+1}{1-x^2}\,dx = \int \left(-1\,\frac{1}{1-x^2}\right) dx$

$$= -x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.$$

4. $\displaystyle\int \tan^2 x\,dx = \int [(1+\tan^2 x) - 1]\,dx$

$$= \int \left(\frac{1}{\cos^2 x} - 1\right) dx = \tan x - x + C.$$

5. $\displaystyle\int \frac{2x+3}{3x+2}\,dx = \int \frac{2\left(x+\frac{3}{2}\right)}{3\left(x+\frac{2}{3}\right)}\,dx$

$$= \frac{2}{3}\int \frac{\left[\left(x+\frac{2}{3}\right)+\frac{5}{6}\right]}{\left(x+\frac{2}{3}\right)}\,dx$$

$$= \frac{2}{3}x + \frac{5}{9}\ln\left|x+\frac{2}{3}\right| + C.$$

6. $\displaystyle\int \sqrt{1-\sin 2x}\,dx$

$$= \int \sqrt{\sin^2 x - 2\sin x\cos x + \cos^2 x}\,dx$$

$$= \int \sqrt{(\sin x - \cos x)^2}\,dx$$

$$= \int |\cos x - \sin x|\,dx$$

$$= (\sin x + \cos x)\cdot \operatorname{sgn}(\cos x - \sin x) + C.$$

III. Problems and Exercises for Independent Work

Find the following integrals:

1. $\displaystyle\int (3-x^2)^3\,dx.$ 2. $\displaystyle\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx.$

3. $\displaystyle\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}\,dx.$ 4. $\displaystyle\int \frac{x^2+3}{x^2-1}\,dx.$

5. $\displaystyle\int \frac{2^{x+1}-5^{x-1}}{10^x}\,dx.$ 6. $\displaystyle\int \tanh^2 x\,dx.$

7. $\displaystyle\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx.$ 8. $\displaystyle\int (2^x+3^x)^2\,dx.$

9. $\displaystyle\int \sqrt{1+\sin 2x}\,dx.$ 10. $\displaystyle\int (2x-3)^{10}\,dx.$

11. $\int \frac{dx}{\sqrt{2-5x}}$. 12. $\int \frac{dx}{2+3x^2}$.

13. $\int \frac{dx}{\sqrt{3x^2-2}}$. 14. $\int \frac{dx}{\sin^2\left(2x+\frac{\pi}{4}\right)}$.

15. $\int \frac{dx}{1+\cos x}$. 16. $\int \frac{dx}{1+\sin x}$.

5.3. The Method of a Change of Variable

I. Fundamental Concepts and Theorems

Theorem 2. *Assume that the function $x = \varphi(t)$ is defined and differentiable on the interval T and the interval X is its range. Let the function $y = f(x)$ be defined on X and have an antiderivative $F(x)$ on it. Then, on the interval T, the function $F(\varphi(t))$ is an antiderivative of the function $f(\varphi(t))\varphi'(t)$.*

It follows from Theorem 2 that

$$\int f(\varphi(t))\varphi'(t)\,dt = F(\varphi(t)) + C, \tag{1}$$

and since $F(\varphi(t))+C=(F(x)+C)\,|_{x=\varphi(t)}=\int f(x)\,dx\Big|_{x=\varphi(t)}$ we can write relation (1) in the form

$$\int f(x)\,dx\,|_{x=\varphi(t)} = \int f(\varphi(t))\varphi'(t)\,dt. \tag{2}$$

Relation (2) is known as the **formula for a change of variable** in an indefinite integral.

If the function $x = \varphi(t)$ has an inverse $t = \varphi^{-1}(x)$, then it follows from relation (2) that

$$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt\,|_{t=\varphi^{-1}(x)}. \tag{3}$$

This is the basic "working" formula used to evaluate the integral $\int f(x)\,dx$ by the method of a change of variable.

II. Control Questions and Assignments

1. Write formula (2) for a change of variable in an indefinite integral. Under what conditions does this formula hold true?

2. Under what condition does formula (2) yield formula (3)?

3. We must find $\int \sqrt{4-x^2}\,dx$ for $-2 \leqslant x \leqslant 2$. Find whether a change of variable is admissible for this purpose in the following cases: (a) $x=\sin t$, $-\pi/2 \leqslant t \leqslant \pi/2$. (b) $x=2\sin t$, $0 \leqslant t \leqslant \pi/2$, (c) $x=2\sin t$, $-\pi/2 \leqslant t \leqslant \pi/2$, (d) $x=2\cos t$, $0 \leqslant t \leqslant \pi/2$, (e) $x=2\cos t$, $\pi \leqslant t \leqslant 2\pi$.

III. Worked Problems

We shall consider some techniques of calculating integrals by means of a change of variable.

1°. An identity transformation of the element of integration with an isolation of the differential of the new integration variable (the simplest change of variable).

1. $$\int \frac{x\,dx}{\sqrt{1-x^2}} = \int \frac{\frac{1}{2}\,d(x^2)}{\sqrt{1-x^2}} = \int \frac{-\frac{1}{2}\,d(1-x^2)}{\sqrt{1-x^2}}$$
$$= -\frac{1}{2}\int (1-x^2)^{-1/2}\,d(1-x^2)$$
$$= -\frac{1}{2}\cdot 2(1-x^2)^{1/2}+C.$$

2. $$\int \frac{x^3\,dx}{x^8-2} = \int \frac{\frac{1}{4}\,d(x^4)}{(x^4)^2-2} = \int \frac{\frac{\sqrt{2}}{4}\,d\left(\frac{x^4}{\sqrt{2}}\right)}{-2\left[1-\left(\frac{x^4}{\sqrt{2}}\right)^2\right]}$$
$$= -\frac{\sqrt{2}}{8}\ln\left|\frac{\sqrt{2}+x^4}{\sqrt{2}-x^4}\right|+C.$$

3. $$\int \frac{dx}{x\sqrt{x^2+1}} = \int \frac{dx}{x^2\sqrt{1+\frac{1}{x^2}}} = \int \frac{-d\left(\frac{1}{x}\right)}{\sqrt{1+\left(\frac{1}{x}\right)^2}}$$
$$= -\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right|+C \quad (x>0).$$

4. $$\int \frac{dx}{x\ln x\ln\ln x} = \int \frac{d(\ln\ln x)}{\ln\ln x} = \ln|\ln\ln x|+C.$$

5. $$\int \tan x\,dx = \int \frac{-d(\cos x)}{\cos x} = -\ln|\cos x|+C.$$

6. $\displaystyle\int \frac{dx}{\sin^2 x+2\cos^2 x} = \int \frac{dx}{\cos^2 x(\tan^2 x+2)}$

$$= \int \frac{\sqrt{2}\,d\left(\frac{\tan x}{\sqrt{2}}\right)}{2\left[1+\left(\frac{\tan x}{\sqrt{2}}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C.$$

2°. Some substitutions.

1. $I = \int x^2 \sqrt[3]{1-x}\,dx$. We set $t=(1-x)^{1/3}$ and then $x=1-t^3$, $dx=-3t^2\,dt$. We have

$$I = \int (1-t^3)^2 t(-3t^2)\,dt = -3\int (1-t^3)^2 t^3\,dt$$

$$= -3\int (t^3-2t^6+t^9)\,dt = -3\left(\frac{t^4}{4}-\frac{2}{7}t^7\right.$$

$$\left.+\frac{1}{10}t^{10}\right)+C,$$

where $t=(1-x)^{1/3}$.

2. $I = \int x^5(2-5x^3)^{2/3}\,dx$. We set $t=2-5x^3$ and then $x=\left(\frac{2-t}{5}\right)^{1/3}$, $dt=-15x^2\,dx$. We find that

$$I = \int \frac{1}{5}(2-t)\,t^{2/3}\left(-\frac{dt}{15}\right) = -\frac{1}{75}(2t^{2/3}-t^{5/3})\,dt$$

$$= -\frac{1}{75}\cdot 2\cdot\frac{3}{5}\cdot t^{5/3} + \frac{1}{75}\cdot\frac{3}{8}t^{8/3}+C,$$

where $t=2-5x^3$.

3. $I = \displaystyle\int \frac{\sin x\cos^3 x}{1+\cos^2 x}\,dx = \int \frac{\sin x\cos x\,[(\cos^2 x+1)-1]}{1+\cos^2 x}\,dx.$

We set $t=1+\cos^2 x$, whence we find that $dt = -2\cos x\sin x\,dx$. This means that

$$I = -\frac{1}{2}\int \frac{t-1}{t}\,dt = -\frac{t}{2}+\frac{1}{2}\ln|t|+C,$$

where $t=1+\cos^2 x$.

3°. Integration of some irrational functions by trigonometric substitutions:

1. $T=\int \frac{\mathrm{d}x}{(1-x^2)^{3/2}}$. We set $x=\sin t$, $-\pi/2<t<\pi/2$ and then $\mathrm{d}x=\cos t\,\mathrm{d}t$. Consequently,

$$I=\int \frac{\cos t\,\mathrm{d}t}{\cos^3 t}=\tan t+C, \quad \text{where } t=\arcsin x.$$

2. $I=\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}}$. We set $x=a\tan t$, $-\pi/2<t<\pi/2$ and then $\mathrm{d}x=\frac{a\,\mathrm{d}t}{\cos^2 t}$ and therefore

$$I=\int \frac{a\cos^3 t\,\mathrm{d}t}{a^3\cos^2 t}=\frac{1}{a^2}\sin t+C, \text{ where } t=\arctan\frac{x}{a}.$$

3. $I=\int \frac{\mathrm{d}x}{(x^2-1)^{3/2}}$. We set $x=1/\sin t$, $-\pi/2<t<0$ and $0<t<\pi/2$ and then $\mathrm{d}x=-\frac{\cos t\,\mathrm{d}t}{\sin^2 t}$. Thus we have

$$I=\int \frac{-\cos t\,\mathrm{d}t}{\sin^2 t\left(\frac{1}{\sin^2 t}-1\right)^{3/2}}=-\int \frac{\sin t\,\mathrm{d}t}{\cos^2 t}$$

$$=\int \frac{\mathrm{d}(\cos t)}{\cos^2 t}=-\frac{1}{\cos t}+C, \text{ where } t=\arcsin\left(\frac{1}{x}\right).$$

IV. Problems and Exercises for Independent Work

Evaluate the following integrals by isolating the differential of the new variable:

1 . $\int x^2\sqrt{1+x^3}\,\mathrm{d}x$. 18. $\int \frac{x\,\mathrm{d}x}{4+x^4}$.

19. $\int \frac{\mathrm{d}x}{(1+x)\sqrt{x}}$. 20. $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$.

21. $\int \frac{\mathrm{d}x}{\sqrt{x}\,(1+x)}$. 22. $\int xe^{-x^2}\,\mathrm{d}x$.

23. $\int \frac{\ln^2 x}{x}\,\mathrm{d}x$. 24. $\int \sin^5 x\cos x\,\mathrm{d}x$.

25. $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\,\mathrm{d}x$. 26. $\int \frac{\sin x\,\mathrm{d}x}{\sqrt{\cos 2x}}$. 27. $\int \frac{\mathrm{d}x}{\sin x}$.

28. $\int \frac{\mathrm{d}x}{\sinh x}$. 29. $\int \frac{\arctan x}{1+x^2}\,\mathrm{d}x$.

30. $\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}\,\mathrm{d}x$.

Evaluate the following integrals by using various substitutions:

31. $\int x^3 (1-5x^2)^{10}\, dx$. 32. $\int \frac{x^5\, dx}{\sqrt{1-x^2}}$.

33. $\int \cos^5 x \sqrt{\sin x}\, dx$. 34. $\int \frac{dx}{e^{x/2}+e^x}$.

35. $\int \frac{dx}{\sqrt{1+e^x}}$. 36. $\int \frac{\arctan \sqrt{x}}{\sqrt{x}} \frac{dx}{1+x}$.

37. $\int \sqrt{a^2-x^2}\, dx$. 38. $\int \sqrt{\frac{a+x}{a-x}}$ $(\alpha x = a \cos 2t)$.

39. $\int \frac{x^2\, dx}{\sqrt{a^2+x^2}}$. 40. $\int \sqrt{\frac{x-a}{x+a}}$ $\left(\alpha x = \frac{a}{\cos 2t}\right)$.

41. $\int \frac{dx}{\sqrt{x^2+a^2}}$. 42. $\int \frac{dx}{\sqrt{x^2-a^2}}$.

5.4. Integration by Parts

I. Fundamental Concepts and Theorems

Theorem 3. *Assume that on the interval X the functions $u(x)$ and $v(x)$ are differentiable and there is $\int v(x)\, u'(x)\, dx$ (i.e. the function $v(x)\, u'(x)$ has an antiderivative on X). Then $\int u(x)\, v'(x)\, dx$ also exists on X and*

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx.$$

This relation is known as the **formula for integration by parts**. Since $u'(x)\, dx = du$, $v'(x)\, dx = dv$, we can write this formula as

$$\int u\, dv = u(x)\, v(x) - \int v\, du.$$

It is convenient to use integration by parts in the following cases.

1. The element of integration contains one of the functions $\ln x$, $\ln \varphi(x)$, $\arcsin x$, $\arccos x$, $\arctan x$ as a factor. If we take one of these functions as $u(x)$, then the element of integration $v\, du$ of the new integral is usually much simpler than the initial one.

2. The integrand function has the form $P(x)\, e^{ax}$, $P(x) \sin ax$, $P(x) \cos ax$, where $P(x)$ is a polynomial with respect to the variable x. If we take $P(x)$ as $u(x)$, then in the new integral the integrand is again of one of the indicated types but the degree of the polynomial is smaller by unity. By choosing this polynomial again as $u(x)$, we lower the degree of the polynomial by unity again etc.

3. The integrand function has the form $e^{ax} \sin bx$, $e^{ax} \cos bx$, $\sin(\ln x)$, $\cos(\ln x)$ etc. After a two-fold integration by parts, we get the initial integral with a coefficient. The relation obtained is a linear algebraic equation with respect to the required integral.

II. Control Questions and Assignments

1. Write the formula for integration by parts for an indefinite integral. Under what conditions does this formula hold true?

2. What functions are convenient for integration by parts?

III. Worked Problems

1. $I = \int \arctan x \, dx$. We set $u = \arctan x$, $dv = dx$. Then $du = \frac{dx}{1+x^2}$, $v = x$. Consequently,

$$I = x \arctan x - \int \frac{x\, dx}{1+x^2} = x \arctan x - \frac{1}{2} \int \frac{d(1+x^2)}{1+x^2}$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

2. $I = \int x^2 e^{-x} \, dx$. We set $u = x^2$, $dv = e^{-x} dx$. Then $du = 2x\, dx$, $v = -e^{-x}$. Hence

$$I = -x^2 e^{-x} + 2 \int x e^{-x} \, dx = -x^2 e^{-x}$$

$$+ 2\left(-x e^{-x} + \int e^{-x} dx\right)$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + C.$$

3. $I = \int \sin(\ln x)\,dx$. We set $u = \sin \ln x$, $dv = dx$. Then $du = \frac{1}{x} \cos \ln x \, dx$, $v = x$. We have

$$I = x \sin \ln x - \int \cos \ln x \, dx = x \sin \ln x - \left(x \cos \ln x + \int \sin \ln x dx \right).$$

We have obtained an equation

$$I = x(\sin \ln x - \cos \ln x) - I,$$

which is linear with respect to I, and find from it that

$$I = \frac{x}{2} (\sin \ln x - \cos \ln x) + C.$$

4. $K_\alpha = \int \frac{dx}{(x^2 + a^2)^\alpha}$ $(\alpha = 1, 2, \ldots)$. We set $u = \frac{1}{(x^2 + a^2)^\alpha}$, $dv = dx$. Then we have

$$\begin{aligned} K_\alpha &= \frac{x}{(x^2+a^2)^\alpha} - \int x d \left(\frac{1}{(x^2+a^2)^\alpha} \right) \\ &= \frac{x}{(x^2+a^2)^\alpha} + 2\alpha \int \frac{(x^2+a^2) - a^2}{(x^2+a^2)^{\alpha+1}} dx \\ &= \frac{x}{(x^2+a^2)^\alpha} + 2\alpha \left[\int \frac{dx}{(x^2+a^2)^\alpha} - a^2 \int \frac{dx}{(x^2+a^2)^{\alpha+1}} \right] \\ &= \frac{x}{(x^2+a^2)^\alpha} + 2\alpha [K_\alpha - a^2 K_{\alpha+1}], \end{aligned}$$

whence

$$K_{\alpha+1} = \frac{1}{2\alpha a^2} \frac{x}{(x^2+a^2)^\alpha} + \frac{2\alpha - 1}{2\alpha a^2} K_\alpha.$$

We have got a recurrent formula by means of which we can express $K_{\alpha+1}$ in terms of K_α. For $\alpha = 1$ the integral K_α is "almost" a tabular integral:

$$K_1 = \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C.$$

Setting $\alpha = 1$ in the recurrent formula and knowing K_1, we can find K_2. Setting $\alpha = 2$ and knowing K_2, we can find K_3 and so on.

Remark. Using the change of variable and integration by parts, we can get the following much needed formulas:

1. $\displaystyle\int \frac{\mathrm{d}x}{a^2+x^2}=\frac{1}{a}\arctan\frac{x}{a}+C \quad (a\neq 0).$

2. $\displaystyle\int \frac{\mathrm{d}x}{a^2-x^2}=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C \quad (a\neq 0).$

3. $\displaystyle\int \frac{x\,\mathrm{d}x}{a^2\pm x^2}=\pm\frac{1}{2}\ln|a^2\pm x^2|+C.$

4. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+C \quad (a>0).$

5. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}=\ln|x+\sqrt{x^2\pm a^2}|+C.$

6. $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{a^2\pm x^2}}=\pm\sqrt{a^2\pm x^2}+C.$

7. $\displaystyle\int \sqrt{a^2-x^2}\,\mathrm{d}x=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C \quad (a>0)$

8. $\displaystyle\int \sqrt{x^2\pm a^2}\,\mathrm{d}x=\frac{x}{2}\sqrt{x^2\pm a^2}\pm\frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}|+C.$

IV. Problems and Exercises for Independent Work

Evaluate each of the following integrals:

43. $\int \ln x\,\mathrm{d}x.$ **44.** $\int \sqrt{x}\ln^2 x\,\mathrm{d}x.$ **45.** $\int x^3 e^{-x^2}\,\mathrm{d}x.$

46. $\int x^2\sin 2x\,\mathrm{d}x.$ **47.** $\int \arcsin x\,\mathrm{d}x.$

48. $\displaystyle\int \frac{\arcsin x}{x^2}\,\mathrm{d}x.$ **49.** $\int \sin x\ln(\tan x)\,\mathrm{d}x.$

50. $\int x(\arctan x)^2\,\mathrm{d}x.$ **51.** $\displaystyle\int \frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,\mathrm{d}x.$

52. $\int \sqrt{x^2\pm a^2}\,\mathrm{d}x.$ **53.** $\int \cos(\ln x)\,\mathrm{d}x.$

54. $\int e^{ax}\sin bx\,\mathrm{d}x.$ **55.** $\int e^{2x}\sin^2 x\,\mathrm{d}x.$

5.5. Integration of Rational Functions

I. Fundamental Concepts and Theorems

1. Decomposition of a rational fraction into the sum of partial fractions. Let us consider a rational function (or rational fraction) $P_n(x)/Q_m(x)$. Here $P_n(x)$ and $Q_m(x)$ are polynomials of degrees n and m with respect to the variable x.

If $n \geqslant m$, i.e. the fraction is improper, we can represent it in the form

$$\frac{P_n(x)}{Q_m(x)} = P_{n-m}(x) + \frac{R_k(x)}{Q_m(x)} \quad (k < m)$$

or, as they say, isolate the integral part $P_{n-m}(x)$ from it.

As a result, the integration of an improper fraction reduces to the integration of a proper fraction $R_k(x)/Q_m(x)$.

Theorem 4. *Assume that $P_n(x)/Q_m(x)$ is a proper rational fraction ($n < m$) and the decomposition $Q_m(x)$ into the product of irreducible real factors has the form*

$$Q_m(x) = (x-a)^{\alpha} \ldots (x-b)^{\beta} (x^2 + px + q)^{\gamma} \ldots (x^2 + rx + s)^{\delta},$$

where $a, \ldots, b$ are real roots and $x^2 + px + q, \ldots, x^2 + rx + s$ are quadratic trinomials irreducible into real factors. Then we have

$$\begin{aligned}\frac{P_n(x)}{Q_m(x)} &= \frac{A_\alpha}{(x-a)^\alpha} + \frac{A_{\alpha-1}}{(x-a)^{\alpha-1}} + \ldots + \frac{A_1}{x-a} \\ &+ \ldots + \frac{B_\beta}{(x-b)^\beta} + \frac{B_{\beta-1}}{(x-b)^{\beta-1}} + \ldots + \frac{B_1}{x-b} \\ &+ \frac{M_\gamma x + N_\gamma}{(x^2+px+q)^\gamma} + \ldots + \frac{M_1 x + N_1}{x^2+px+q} \\ &+ \ldots + \frac{K_\delta x + L_\delta}{(x^2+rx+s)^\delta} + \ldots + \frac{K_1 x + L_1}{x^2+rx+s},\end{aligned} \tag{1}$$

where A_i, B_i, M_i, N_i, K_i, and L_i are real numbers.

The fractions which appear on the right-hand side of (1) are *partial fractions* and relation (1) is the decomposition of a proper rational fraction $P_n(x)/Q_m(x)$ into a sum of partial fractions with real coefficients.

2. Integrability of partial fractions in terms of elementary functions. Each of the partial fractions can be in-

tegrated in terms of elementary functions:

(1) $\int \frac{A}{x-a}\,dx = A \ln|x-a| + C$,

(2) $\int \frac{B}{(x-a)^\alpha}\,dx = \frac{B}{1-\alpha}\frac{1}{(x-a)^{\alpha-1}} + C \quad (\alpha > 1)$,

(3) $$\int \frac{Mx+N}{x^2+px+q}\,dx = \frac{M}{2}\ln(x^2+px+q) + \frac{2N-Mp}{2\sqrt{q-\frac{p^2}{4}}}\arctan\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C,$$

(4) $$\int \frac{Mx+N}{(x^2+px+q)^\alpha}\,dx = \frac{M}{2(1-\alpha)}\frac{1}{(t^2+a^2)^{\alpha-1}} + \left(N-\frac{Mp}{2}\right)K_\alpha \quad (\alpha > 1),$$

where

$$K_\alpha = \int \frac{dt}{(t^2+a^2)^\alpha}, \quad t = x+\frac{p}{2}, \quad a^2 = q-\frac{p^2}{4}.$$

The integral K_α can be found from the recurrent formula (see 5.4).

3. Techniques of decomposition of a proper fraction into a sum of partial fractions.

Method of undetermined coefficients. We shall use the following example to get acquainted with this method. The decomposition of the proper fraction $\frac{x}{(x+1)(2x-1)(x^2+1)}$ into a sum of partial fractions in accordance with Theorem 4 has the form

$$\frac{x}{(x+1)(2x-1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{2x-1} + \frac{Cx+D}{x^2+1}.$$

Reducing to a common denominator and equating the numerators of the resulting fractions, we arrive at a relation

$$x = A(2x-1)(x^2+1) + B(x+1)(x^2+1) + Cx(x+1)(2x-1) + D(x+1)(2x-1). \quad (2)$$

Two polynomials are identically equal if and only if the coefficients in the same powers of x are equal. Equating the coefficients to one another, we get a system of

equations for A, B, C, and D:

$$\begin{aligned}
&\text{for } x^0: && -A + B - && D = 0,\\
&\text{for } x^1: && 2A + B - C + && D = 1,\\
&\text{for } x^2: && -A + B + C + && 2D = 0,\\
&\text{for } x^3: && 2A + B + 2C && = 0.
\end{aligned}$$

Solving this system, we find the coefficients A, B, C and D.

Even in the example involving four undetermined coefficients, such a general approach leads to a rather cumbersome system.

Using identity (2), we can find the required coefficients much easier:

(a) setting $x=\frac{1}{2}$ in (2), we get $\frac{1}{2}=B\frac{3}{2}\cdot\frac{5}{4}$, whence $B=\frac{4}{15}$,

(b) setting $x=-1$ in (2), we get $-1=A\cdot(-3)\cdot 2$, whence $A=\frac{1}{6}$,

(c) setting $x=0$ in (2), we get $0=-A+B-D$, whence $D=B-A=\frac{1}{10}$,

(d) comparing the coefficients in x^3, we get $0=2A+B+2C$, whence $C=-\frac{1}{2}(2A+B)=-\frac{1}{2}\left(\frac{1}{3}+\frac{4}{15}\right)=-\frac{3}{10}$.

Method of cancelling. Here is a useful technique used to calculate some undetermined coefficients. Assume that

$$\frac{P_n(x)}{Q_m(x)}=\frac{P_n(x)}{(x-a)^{\alpha}\varphi(x)}, \quad \text{where } \varphi(a)\neq 0,$$

i.e. the real number a is a root of multiplicity α of the polynomial $Q_m(x)$.

In the decomposition of the fraction $P_n(x)/Q_m(x)$, the chain of partial fractions is associated with the multiple real root $x=a$:

$$\frac{A_{\alpha}}{(x-a)^{\alpha}}+\frac{A_{\alpha-1}}{(x-a)^{\alpha-1}}+\ldots+\frac{A_1}{x-a}.$$

To find the coefficient A_{α} in the leading power of the denominator, we must cancel $(x-a)^{\alpha}$ in the deno-

minator of the initial fraction $\frac{P_n(x)}{(x-a)^{\alpha}\varphi(x)}$ and set $x=a$ in the remaining fraction, i. e. $A_\alpha=\frac{P_n(a)}{\varphi(a)}$.

This method is especially convenient when all the roots of the denominator are real and simple. Then this method is used to find all undetermined coefficients. For example,

$$\frac{x+2}{x(x-1)(x+1)(x-2)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}+\frac{D}{x-2},$$

$$A=\left.\frac{x+2}{(x-1)(x+1)(x-2)}\right|_{x=0}=1,$$

$$B=\left.\frac{x+2}{x(x+1)(x-2)}\right|_{x=1}=-\frac{3}{2}\ \text{etc.}$$

II. Control Questions and Assignments

1. Is every rational fraction integrable in terms of elementary functions?
2. Why do we use the test for integrability only for proper fractions?
3. What does it mean "to isolate the integral part of an improper fraction"?
4. Into what prime real factors can a polynomial with real coefficients be decomposed?
5. The number $2-i$ is known to be a root of a polynomial with real coefficients. Is it true that the number $2+i$ is a root of the same polynomial?
6. Into what partial fractions can the fraction $\frac{x+1}{(x+1)^2(x^2+x+1)}$ be decomposed?
7. What is the method of undetermined coefficients used for decomposing a fraction into a sum of partial fractions?
8. What is the method of cancellation in computing undetermined coefficients?
9. Use the method of cancellation to find the undetermined coefficients in the decomposition of the fraction $\frac{x}{(x+2)(x-3)}$.
10. Use the method of cancellation to find undetermined coefficients in the decomposition of the fraction $\frac{x^2}{(x^2-2)(x^2+3)}$. ● Set $x^2=y$ and then the method of cancellation.

III. Problems and Exercises for Independent Work

Find each of the following integrals:

56. $\int \frac{2x+3}{(x-2)(x+5)}\,dx$. 57. $\int \frac{x^3+1}{x^3-5x^2+6x}\,dx$.

58. $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx$. 59. $\int \frac{x^2+5x+4}{x^4+5x^2+4}\,dx$.

60. $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)}$. 61. $\int \frac{dx}{x^3+1}$.

62. $\int \frac{dx}{x^4-1}$. 63. $\int \frac{dx}{x^4+1}$. 64. $\int \frac{dx}{x^4+x^2+1}$.

65. $\int \frac{x^2\,dx}{x^4+3x^3+4.5x^2+3x+1}$. 66. $\int \frac{dx}{x^6+1}$.

67. $\int \frac{x^3\,dx}{(x-1)^{100}}$. 68. $\int \frac{x^2+x}{x^6+1}\,dx$.

69. $\int \frac{x^{11}\,dx}{x^8+3x^4+2}$. 70. $\int \frac{1-x^7}{x(1+x^7)}\,dx$.

71. $\int \frac{x^2+1}{x^4+x^2+1}\,dx$. 72. $\int \frac{x^5-x}{x^8+1}\,dx$.

73. $\int \frac{x^4+1}{x^6+1}\,dx$.

5.6. Integration of Irrational Functions

I. Fundamental Rationalizing Substitutions

Here and in the following sections we designate a rational function of two arguments x and y as $R(x, y)$.

1. **Integration of linear fractional irrationalities.** The integral of the form $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$ $\left(\frac{a}{c} \neq \frac{b}{d}\right)$ can be rationalized, i.e. reduced to an integral of a rational function, by means of the substitution

$$t = \sqrt[n]{\frac{ax+b}{cx+d}}.$$

2. **Euler's substitutions.** Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})\,dx$ $(a \neq 0)$ can be integrated in terms of elementary functions by means of Euler's substitutions which rationalize integrals of this form.

If the quadratic trionomial ax^2+bx+c has complex roots (in that case the sign of a coincides with that of the

trinomial which appears under the radical sign, i.e. $a > 0$) then we use the *first Euler's substitution*:

$$t = \sqrt{ax^2 + bx + c} + x\sqrt{a}\,.$$

If $ax^2 + bx + c = a(x - x_1)(x - x_2)$, where x_1 and x_2 are real roots, then we use the *second Euler's substitution* to rationalize the integral:

$$t = \frac{\sqrt{ax^2 + bx + c}}{x - x_1}\,.$$

3. Other techniques of integration of quadratic irrationalities. Although they play an important theoretical part, in practical applications Euler's substitutions usually lead to complicated computations and are therefore only used when the integral cannot be calculated by any other, simpler, technique. One of the simpler tecnniques is the following. If we isolate a perfect square in the quadratic trinomial $ax^2 + bx + c$, i.e. reduce it o the form $a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$, and set

$$t = \sqrt{\frac{a}{\left|c - \frac{b^2}{4a}\right|}}\left(x + \frac{b}{2a}\right),$$

then the integral $\int R(x, \sqrt{ax^2 + bx + c})\,dx$ reduces to tone of the following three forms:

$$\int R_1(t, \sqrt{1 - t^2})\,dt, \quad \int R_2(t, \sqrt{t^2 - 1})\,dt,$$
$$\int R_3(t, \sqrt{1 + t^2})\,dt.$$

Making the substitution $t = \sin u$ in the first integral, $1/t = \sin u$ in the second and $t = \tan u$ in the third, we get integrals of the form $\int R(\sin u, \cos u)\,du$ (see 5.7).

Two special cases must be pointed out.

1. For an integral of the form $\int \frac{P_n(x)}{\sqrt{ax^2 + bx + c}}\,dx$, where $P_n(x)$ is a polynomial, there holds a formula

$$\int \frac{P_n(x)\,dx}{\sqrt{ax^2 + bx + c}} = Q_{n-1}(x)\sqrt{ax^2 + bx + c} + \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}}\,, \tag{1}$$

where $Q_{n-1}(x)$ is a polynomial of degree $n-1$ and λ is a number. To find the coefficients in $Q_{n-1}(x)$ and the number λ, we differentiate identity (1). Reducing then to a common denominator, we get an equality of two polynomials. Comparing the coefficients in the same powers of x, we find the unknown coefficients. The integral which enters into the right-hand side of relation (1) can be calculated by the indicated technique by means of one of the trigonometric substitutions.

2. An integral of the form $\int \frac{dx}{(x-\beta)^m \sqrt{ax^2+bx+c}}$ can be reduced to the preceding one by means of the substitution $t=1/(x-\beta)$.

II. Control Questions and Assignments

1. What substitution rationalizes an integral of a linear fractional irrationality?
2. Integrals of what form can be calculated by means of Euler's substitutions?
3. What is the theoretical part played by Euler's substitutions?
4. By means of what trigonometric substitutions can the following integrals be calculated:

$$\int \sqrt{1-x^2}\,dx, \quad \int \sqrt{x^2-3}\,dx, \quad \int \sqrt{x^2+3}\,dx,$$

$$\int \sqrt{x^2-2x+2}\,dx, \quad \int \sqrt{2+2x-x^2}\,dx?$$

III. Worked Problems

1. $I=\int \sqrt[3]{\frac{x+1}{x-1}}\,\frac{dx}{x+1}$. We set $t=\sqrt[3]{\frac{x+1}{x-1}}$, whence we find that $dx=-\frac{6t^2\,dt}{(t^3-1)^2}$. Consequently,

$$I=-3\int \frac{dt}{t^3-1}=-\frac{1}{2}\ln\frac{(1-t)^2}{1+t+t^2}$$

$$+\sqrt{3}\arctan\frac{2t+1}{\sqrt{3}}+C,$$

where $t=\sqrt[3]{\frac{x+1}{x-1}}$.

2. $I = \int \frac{dx}{(x^2+x+1)\sqrt{x^2+x-1}}$

$$= \int \frac{dx}{\left[\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right]\sqrt{\left(x+\frac{1}{2}\right)^2-\frac{5}{4}}}$$

$$= \int \frac{dx}{\frac{5\sqrt{5}}{4\cdot 2}\left[\left(\frac{2x+1}{\sqrt{5}}\right)^2+\frac{3}{5}\right]\sqrt{\left(\frac{2x+1}{\sqrt{5}}\right)^2-1}}.$$

We set $t=\frac{2x+1}{\sqrt{5}}$, and then $\sqrt{5}\,dt=2dx$ and, hence,

$$I=\frac{4}{5}\int \frac{dt}{\left(t^2+\frac{3}{5}\right)\sqrt{t^2-1}}.$$

Now we set $\frac{1}{t}=\sin u$, whence $-\frac{dt}{t^2}=\cos u\,du$. Therefore

$$I=-\frac{4}{5}\int \frac{\cos u\,du\sin u}{\left(1+\frac{3}{5}\sin^2 u\right)\cos u}$$

$$=-\frac{4}{5}\int \frac{\sin u\,du}{\frac{8}{5}-\frac{3}{5}\cos^2 u}=\frac{4}{3}\int \frac{d(\cos u)}{\frac{8}{3}-\cos^2 u}$$

$$=\frac{4}{3}\,\frac{1}{2}\sqrt{\frac{3}{8}}\ln\left|\frac{\sqrt{8}+\sqrt{3}\cos u}{\sqrt{8}-\sqrt{3}\cos u}\right|+C,$$

where $u=\arcsin\frac{\sqrt{5}}{2x+1}$. The final result is

$$I=\frac{1}{\sqrt{6}}\ln\left|\frac{(2x+1)\sqrt{2}+\sqrt{3(x^2+x-1)}}{(2x+1)\sqrt{2}-\sqrt{3(x^2+x-1)}}\right|+C.$$

3. $I=\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$. We set $t=\frac{1}{x-1}$ and then $dx=-\frac{1}{t^2}dt$ and $I=-\int \frac{t^2\,dt}{\sqrt{5t^2+5t+1}}$. Next we use formula (1):

$$\int \frac{t^2\,dt}{\sqrt{5t^2+5t+1}}=(At+B)\sqrt{5t^2+5t+1}$$
$$+\lambda\int \frac{dt}{\sqrt{5t^2+5t+1}}.$$

Differentiating this identity, we obtain

$$\frac{t^2}{\sqrt{5t^2+5t+1}} = A\sqrt{5t^2+5t+1} + \frac{(At+B)(10t+5)}{2\sqrt{5t^2+5t+1}} + \frac{\lambda}{\sqrt{5t^2+5t+1}}.$$

Reducing to a common denominator and comparing the coefficients in the same powers of t, we find that $A=1/10$, $B=-3/20$, $\lambda=11/40$. Furthermore,

$$\int \frac{dt}{\sqrt{5t^2+5t+1}} = \frac{1}{\sqrt{5}} \int \frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2-\frac{1}{20}}$$

$$= \frac{1}{\sqrt{5}} \ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{5}}\right| + C.$$

The final result is

$$I = -\left(\frac{1}{10}t-\frac{3}{20}\right)\sqrt{5t^2+5t+1} - \frac{11}{40\sqrt{5}} \ln\left|t+\frac{1}{2} + \sqrt{t^2+t+\frac{1}{5}}\right| + C,$$

where $t=\frac{1}{x-1}$, or

$$I = \frac{3x-5}{20(x-1)^2}\sqrt{x^2+3x+1} - \frac{11}{40\sqrt{5}} \ln\left|\frac{(x+1)\sqrt{5}+2\sqrt{x^2+3x+1}}{x-1}\right| + C.$$

IV. Problems and Exercises for Independent Work

Find each of the following integrals:

74. $\int \frac{dx}{x(1+2\sqrt{x}+\sqrt[3]{x})}$. **75.** $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,dx$.

76. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$. **77.** $\int \frac{x^2\,dx}{\sqrt{x^2+x+1}}$.

78. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$. **79.** $\int \frac{\sqrt{x^2+2x+2}}{x}\,dx$.

80. $\int \frac{x^3\,dx}{\sqrt{1+2x-x^2}}$. **81.** $\int \frac{x^{10}\,dx}{\sqrt{1+x^2}}$.

82. $\int \frac{dx}{x^3\sqrt{x^2+1}}$. 83. $\int \frac{dx}{x^4\sqrt{x^2-1}}$.

84. $\int \frac{x^2\,dx}{(4-2x+x^2)\sqrt{2+2x-x^2}}$.

85. $\int \frac{dx}{(1-x^4)\sqrt{1+x^2}}$.

86. $\int \frac{x\,dx}{(x^2-1)\sqrt{x^2-x-1}}$.

87. $\int \frac{(x+1)\,dx}{(x^2+x+1)\sqrt{x^2+x+1}}$.

88. $\int \frac{dx}{x+\sqrt{x^2+x+1}}$. 89. $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$.

90. $\int \frac{dx}{[x+\sqrt{x(1+x)}]^2}$. 91. $\int \frac{(x^2-1)\,dx}{(x^2+1)\sqrt{x^4+1}}$.

92. $\int \frac{(x^2+1)\,dx}{(x^2-1)\sqrt{x^4+1}}$.

5.7. Integration of Trigonometric Functions

I. Fundamental Rationalizing Substitutions

An integral of the form $\int R(\sin x, \cos x)\,dx$ can be rationalized by means of a universal trigonometric substitution $t = \tan(x/2)$. In practical applications, it often leads to cumbersome calculations. In a number of cases the following substitutions are more convenient:

(a) $t = \cos x$, if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$,

(b) $t = \sin x$, if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$,

(c) $t = \tan x$, if $R(-\sin x, -\cos x) = R(\sin x, \cos x)$.

II. Worked Problems

1. $I = \int \cos^5 x\,dx$. The integrand function refers to case (b), and therefore we set $t = \sin x$. Then $dt = \cos x\,dx$, $\cos^4 x = (1-\sin^2 x)^2 = (1-t^2)^2$ and

$$I = \int (1-t^2)^2\,dt = t - \frac{2}{3}t^3 + \frac{1}{5}t^5 + C,$$

where $t = \sin x$.

2. $I = \int \sin 5x \cos x \, dx$. In this case it is easier to calculate the integral without resorting to substitutions but representing the integrand as $\frac{1}{2}(\sin 4x + \sin 6x)$. Then

$$I = \frac{1}{2} \int (\sin 4x + \sin 6x) \, dx = -\frac{1}{8} \cos 4x - \frac{1}{12} \cos 6x + C.$$

3. $I = \int \frac{dx}{a \cos x + b \sin x}$. Here we can make a universal substitution $t = \tan \frac{x}{2}$. Then $x = 2 \arctan t$, $dx = \frac{2\,dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and then the integral I reduces to an integral of a rational function. It is easier, however, to begin with the transformation of the integrand

$$\frac{1}{a \cos x + b \sin x} = \frac{1}{\sqrt{a^2+b^2} \sin (x+\varphi)},$$

where $\sin \varphi = \frac{a}{\sqrt{a^2+b^2}}$, $\cos \varphi = \frac{b}{\sqrt{a^2+b^2}}$, and then set $t = \tan \frac{x+\varphi}{2}$. Then $dx = \frac{2\,dt}{1+t^2}$, $\sin (x+\varphi) = \frac{2t}{1+t^2}$ and, consequently,

$$I = \frac{1}{\sqrt{a^2+b^2}} \int \frac{dt}{t} \Big|_{t=\tan\frac{x+\varphi}{2}}$$

$$= \frac{1}{\sqrt{a^2+b^2}} \ln \left| \tan \frac{x+\varphi}{2} \right| + C.$$

III. Problems and Exercises for Independent Work

Find each of the following integrals:

93. $\int \sin^6 x \, dx$. 94. $\int \sin^2 x \cos^4 x \, dx$.

95. $\int \frac{\sin^3 x}{\cos^4 x} \, dx$. 96. $\int \tan^5 x \, dx$.

97. $\int \frac{dx}{\sqrt{\sin^3 x \cos^5 x}}$. 98. $\int \frac{dx}{\sqrt{\tan x}}$.

99. $\int \cos x \cos 2x \cos 3x \, dx$. 100. $\int \sin^3 2x \cos^2 3x \, dx$.

101. $\int \frac{dx}{2 \sin x - \cos x + 5}$.

Chapter 6

Fundamental Theorems on Continuous and Differentiable Functions

6.1. Theorems on the Boundedness of Continuous Functions

I. Fundamental Concepts and Theorems

1. Definition of a bounded function. Let the function $y = f(x)$ be defined on a set X.

Definition. *The function $y = f(x)$ is* ***bounded from above (from below)*** *on a set X if $\exists$ a number M (m) such that $\forall x \in X$ the inequality $f(x) \leqslant M$ $(f(x) \geqslant m)$ holds true.*

The number M (m) is the *upper* (*lower*) *bound* of the function on the set X. The function $y = f(x)$ is *bounded on the set X* (or bounded from both sides) if it is bounded both from above and from below on this set.

2. Theorems on the boundedness of continuous functions.

Theorem 1 (on the local boundedness of a function continuous at a point). *If the function $y = f(x)$ is continuous at a point x_0, then there is a neighbourhood of the point x_0 in which the function is bounded.*

Theorem 2 (on the constancy of sign of a continuous function). *If the function $y = f(x)$ is continuous at a point x_0 and $f(x_0) \neq 0$, then there is a neighbourhood of the point x_0 in which $f(x)$ is of the same sign as $f(x_0)$.*

Theorem 3 (the first Weierstrass theorem). *A function continuous on a closed interval is bounded on that interval.*

3. The least upper bound and the greatest lower bound of a function.

Definition. *The number M is the* ***least upper bound*** *of the function $y = f(x)$ on a set X if*

(1°) $\forall x \in X$ the inequality $f(x) \leqslant M$ holds true,

(2°) $\forall M' < M \ \exists\, x' \in X$ such that $f(x') > M'$.

Remark. Condition 1° signifies that the number M is one of the upper bounds of the function $y = f(x)$ on the set X. Condition 2° signifies that M is the least of the upper bounds of the function $y = f(x)$ on the set X, i.e. there is no number M', smaller than M, which is an upper bound.

The least upper bound of the function $y = f(x)$ on the set X is designated as $\sup\limits_X f(x)$. If the function $y = f(x)$ is unbounded from above on the set X, then we write $\sup\limits_X f(x) = +\infty$.

The greatest lower bound, $\inf\limits_X f(x)$, is defined by analogy. The difference $\sup\limits_X f(x) - \inf\limits_X f(x)$ is the *oscillation* of the function $y = f(x)$ on the set X.

Theorem 4 (the second Weierstrass theorem). *A function $f(x)$ continuous on the closed interval $[a, b]$ attains its least upper and greatest lower bounds on this interval,* i.e. $\exists x'$ *and* $x'' \in [a, b]$ *such that* $f(x') = \inf\limits_{[a,\, b]} f(x)$, $f(x'') = \sup\limits_{[a,\, b]} f(x)$.

If the function $y = f(x)$ attains its least upper (greatest lower) bound on a set X, then it has its *maximum* (*minimum*) *value* on X, and $\max\limits_X f(x) = \sup\limits_X f(x)$ [$\min\limits_X f(x) = \inf\limits_X f(x)$ respectively], otherwise the function does not have a maximum (minimum) value on the set X.

II. Control Questions and Assignments

1. Give a definition of a function bounded from above (from below) on the set X.
2. Using the rule of constructing negations of propositions with quantifiers, formulate the definition of a function unbounded from above (from below) on the set X.
3. Prove that the definition of a bounded function is equivalent to the following definition: the function $y = f(x)$ is bounded on a set X if $\exists$ a number $A > 0$ such that $\forall x \in X$ the inequality $|f(x)| \leqslant A$ holds true.

4. Formulate the definition of a function unbounded on the set X using the negation of the definition given in assignment 3.

5. Formulate the theorem on the local boundedness of a continuous function.

6. Prove that the theorem on the local boundedness of a function remains true if the condition of the continuity of the function at the point x_0 is replaced by the condition of the existence of a limit $\lim\limits_{x \to x_0} f(x)$.

7. Formulate the theorem on the constancy of sign of a continuous function.

8. The function $f(x)$ is known to be continuous at the point x_0 and $f(x_0) = 0$. Can we state that $f(x)$: (a) has a definite sign in a neighbourhood of the point x_0 (except for the point x_0 itself)? (b) does not have a definite sign in any neighbourhood of the point x_0?

9. Formulate the first Weierstrass theorem.

10. Find out whether the following statement is true: "A function continuous on an interval is bounded on that interval".

11. Can a function unbounded on a set X be continuous on this set if: (a) X is a closed interval, (b) X is an open interval?

12. Give a definition of the least upper bound and the greatest lower bound of a function. In what case do we set $\inf\limits_{X} f(x) = -\infty$?

13. Find out whether the following statement is true: "A function bounded from above (from below) on a set X has the least upper (greatest lower) bound on this set".

14. Formulate the second Weierstrass theorem.

15. Find out whether the following statement is true: "A function continuous and bounded on an open interval attains its least upper and greatest lower bounds on that interval".

16. Find out whether the following statement is true: "If a function does not attain its least upper (greatest lower) bound on the closed interval $[a, b]$, then it is discontinuous on this interval".

17. Find out whether the following statement is true: "A function discontinuous on the closed interval $[a, b]$ does not attain its least upper and greatest lower bounds on this interval".

18. Find out whether each of the following statements is true:

(a) "A function $y = f(x)$ bounded on the closed interval $[a, b]$ has $\max\limits_{[a, b]} f(x)$ and $\min\limits_{[a, b]} (f(x)$".

(b) "A function continuous on the closed interval $[a, b]$ has $\max\limits_{[a, b]} f(x)$ and $\min\limits_{[a, b]} f(x)$".

19. Find out whether statements 18 are true if the closed interval $[a, b]$ is replaced by the open interval (a, b).

III. Worked Problems

1. Prove that the function $y = \frac{1}{1+x^2}$ is bounded on the number line $(-\infty, +\infty)$.

△ Since $x^2 \geqslant 0$, we have $1 + x^2 \geqslant 1$ and, consequently, $\forall x \in (-\infty, +\infty)$ there hold inequalities

$$0 \leqslant \frac{1}{1+x^2} \leqslant 1. \tag{1}$$

It follows that the function $y = \frac{1}{1+x^2}$ is bounded on $(-\infty, +\infty)$. ▲

2. Find the least upper and the greatest lower bound of the function given in Example 1 and find out whether it possesses a maximum and a minimum value.

△ It is clear that $\frac{1}{1+x^2} \to 0$ as $x \to \infty$, and therefore this function can assume values arbitrarily close to zero. It is natural to assume that $\inf\limits_{(-\infty, +\infty)} \frac{1}{1+x^2} = 0$. Let us prove this using the definition of the greatest lower bound. We must show that the following two conditions are fulfilled:

(1^0) $\forall x \in (-\infty, +\infty)$: $\frac{1}{1+x^2} \geqslant 0$,

(2^0) $\forall m > 0 \; \exists x'$ such that $\frac{1}{1+x'^2} < m$.

Condition 1^0 is fulfilled by virtue of the left inequality (1). We shall prove that condition 2^0 is also fulfilled. We specify an arbitrary $m > 0$. If $m \geqslant 1$, then the inequality $\frac{1}{1+x'^2} < m$ is fulfilled by virtue of (1)

for all x'. Let $m<1$. Then $\frac{1}{m}-1>0$. We take, as x', any number such that $x'>\sqrt{\frac{1}{m}-1}$. Then $x'^2>\frac{1}{m}-1$ whence $1+x'^2>\frac{1}{m}$ and, consequently, $\frac{1}{1+x'^2}<m$, and this is what we wished to prove. Thus $\inf\limits_{(-\infty,\ +\infty)} \frac{1}{1+x^2}=0$.

We can prove by analogy that $\sup\limits_{(-\infty,\ +\infty)} \frac{1}{1+x^2}=1$.

We take note now of the fact that the function $y=\frac{1}{1+x^2}$ assumes a value of 1 (for $x=0$) but does not assume a value of 0 for any x $\left(\frac{1}{1+x^2}>0\ \forall x\right)$. Therefore this function has a maximum value $\left(\max\limits_{(-\infty,\ +\infty)} \frac{1}{1+x^2}=1\right)$ on $(-\infty,\ +\infty)$ but does not have a minimum value. ▲

IV. Problems and Exercises for Independent Work

1. Prove the boundedness of the function:

(a) $y=\frac{1+x}{1+x^2}$ on the half-open interval $[0,\ +\infty)$,

(b) $y=\frac{2x}{1+x^2}$ on the number line $(-\infty,\ +\infty)$,

(c) $y=x\sin\frac{1}{x}$ on $(-\infty,\ +\infty)$,

(d) $y=\arctan 2^x$ on $(-\infty,\ +\infty)$,

(e) $y=xe^{-x}$ on $(0,\ +\infty)$.

2. Find out whether the following functions are bounded:

(a) $y=x^2$ on $[-5,\ 10]$, (b) $y=x^2$ on $[-5,\ +\infty)$,

(c) $y=x\cos(1/x)$ on $(-\infty,\ +\infty)$,

(d) $y=\begin{cases} 2^{1/(x-1)} & \text{for } x\neq 1, \\ 0 & \text{for } x=1 \end{cases}$ on $(0, 2)$,

(e) $y=2^{1/(x-1)}$ on $(0,\ 1)$.

3. Give an example of a function which, on a set X: (a) is bounded from above and unbounded from below, (b) is bounded from below and unbounded from above, (c) is unbounded both from above and from below.

4. Let the function $f(x)$ be defined on a set X and assume that $\forall x \in X$ there is a neighbourhood in which $f(x)$ is bounded. Does it follow that $f(x)$ is bounded on X if: (a) X is an open interval, (b) X is a closed interval?

5. Give an example of a function $f(x)$ which is continuous and equal to zero at a point x_0 and: (a) is of a definite sign in a neighbourhood of the point x_0 (except for the point x_0 itself), (b) does not retain sign in any neighbourhood of the point x_0.

6. Give an example of a function which is continuous on an open interval but is not bounded on it: (a) from above, (b) from below, (c) from both sides.

7. Give an example of a function defined on the interval $[a, b]$ but unbounded on it: (a) from above, (b) from below, (c) from both sides. Can such a function be continuous on $[a, b]$?

8. Find the least upper and the greatest lower bound of the following functions:

(a) $f(x) = \dfrac{2x}{1+x^2}$ on $(0, +\infty)$,

(b) $f(x) = x^2$ on $[-5, 10]$,

(c) $f(x) = \arctan 2^x$ on $(-\infty, +\infty)$,

(d) $f(x) = \sin x + \cos x$ on $[0, \pi]$,

(e) $f(x) = 2^{1/(x-1)}$ on $(0, 1)$.

Does $f(x)$ attain its least upper and greatest lower bounds on the indicated set?

9. Give an example of a function $f(x)$ which has (a) $\sup\limits_X f(x) = +\infty$, (b) $\inf\limits_X f(x) = -\infty$.

10. Give an example of a function continuous and bounded on an open interval which (a) attains its limit superior but does not attain its limit inferior, (b) attains its limit inferior but does not attain its limit superior, (c) does not attain either the limit superior or the limit interior.

11. Give an example of a function bounded on a closed interval which, on that interval, (a) attains its limit superior but does not attain its limit inferior, (b) attains its limit inferior but does not attain its limit superior,

(c) does not attain either the limit superior or the limit inferior. Can such a function be continuous on a closed interval?

12. Give an example of a function which has both the limit superior and the limit inferior on a set X but does not possess either max or min.

13. Find the oscillations of the following functions:

(a) $f(x) = x^2$ on $(-1, 2)$,

(b) $f(x) = \sin(1/x)$ on $(0, \varepsilon)$, where ε is an arbitrary number,

(c) $f(x) = x \sin(1/x)$ on $(0, 1)$,

(d) $f(x) = x \mid \sin(1/x) \mid$ on $(0, 1)$.

14. We designate the greatest lower and the least upper bound of the function $f(x)$ on a set X as $m[f]$ and $M[f]$ respectively. Let $f_1(x)$ and $f_2(x)$ be defined and bounded on X. Prove that

$$m[f_1 + f_2] \geqslant m[f_1] + m[f_2], \quad M[f_1 + f_2] \leqslant M[f_1] + M[f_2].$$

Give examples of the functions $f_1(x)$ and $f_2(x)$ for which, in the indicated relations, we have (a) the equality sign, (b) the inequality sign.

6.2. Uniform Continuity of a Function

I. Fundamental Concepts and Theorems

1. Definition of the uniform continuity of a function. Assume that a set X is an interval or consists of several intervals.

Definition. *The function $f(x)$ is* ***uniformly continuous*** *on the set X if $\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ such that $\forall x, x' \in X$, which satisfy the inequality $|x - x'| < \delta$ the inequality $|f(x) - f(x')| < \varepsilon$ is satisfied.*

Remark. It follows from the definition that if a function is uniformly continuous on a set X then it is continuous on that set, i.e. continuous at its every point. The difference between the uniform continuity of a function on a set X and the "ordinary" continuity on that set (i.e. continuity at its every point) is that when the continuity is uniform $\forall \varepsilon > 0$ there is a "necessary" (such that is required by the definition) $\delta(\varepsilon) > 0$ common for all $x \in X$ (δ depends only on ε and does not depend on x), whereas when the continuity is "ordinary",

$\forall\varepsilon > 0$ and $\forall x \in X$ there is a "necessary" δ (i.e. δ depends both on ε and on x), but it may so happen that the "necessary" $\delta(\varepsilon) > 0$ common for all $x \in X$ may not exist for some ε. It is clear that in that case δ varies with x so that (for the indicated fixed values of ε) it can assume arbitrarily small values.

2. Geometrical illustration of the uniform continuity of a function. If $f(x)$ is uniformly continuous on X,

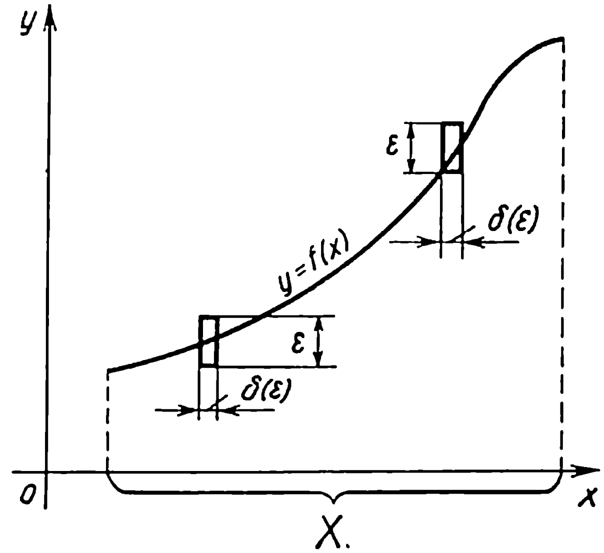


Fig. 5

then $\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0$ such that a rectangle with sides $\delta(\varepsilon)$ and ε, which are parallel to the Ox and Oy axes, can be displaced along the graph (the parallelism of the sides to the coordinate axes being retained) so that the graph will not cut the horizontal sides of the rectangle but will cut its vertical sides (Fig. 5).

3. Theorems on the uniform continuity of a function.

Theorem 5 (Cantor's theorem). *A function continuous on a closed interval is uniformly continuous on that interval.*

Theorem 6 (a sufficient condition for the uniform continuity of a function). *If the function $f(x)$ has a bounded derivative on an interval X, then $f(x)$ is uniformly continuous on that interval.*

II. Control Questions and Assignments

1. Give a definition of the uniform continuity of a function.
2. Using the quantifiers, formulate the negation of the uniform continuity of a function.

3. Find out whether each of the following statements is true:

(a) "If $f(x)$ is continuous on a set X, then it is uniformly continuous on that set".

(b) "If $f(x)$ is uniformly continuous on X, then it is continuous on X".

4. What is the geometrical illustration of the uniform continuity of a function?

5. Formulate Cantor's theorem.

6. Find out whether the following statement is true: "A function which is continuous on an open interval is uniformly continuous on that interval".

7. Formulate the theorem which expresses a sufficient condition for the uniform continuity of a function.

8. Is the boundedness of a derivative the necessary condition for the uniform continuity of a function?

III. Worked Problems

1. Test the function $y = x^2$ for the uniform continuity on the interval $(-l, l)$, where $l > 0$ is any fixed number.

△ We shall prove that the function $y = x^2$ is uniformly continuous on the interval $(-l, l)$ employing three techniques: (1) using the definition of the uniform continuity, (2) using Cantor's theorem, (3) using the sufficient condition for the uniform continuity.

1st technique. We set up a difference $y(x_1) - y(x_2)$:

$$y(x_1) - y(x_2) = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2). \qquad (1)$$

If $x_1, x_2 \in (-l, l)$ then the absolute value of the sum $|x_1 + x_2|$ is bounded by the number $2l$. Therefore the absolute value of the difference $|y(x_1) - y(x_2)|$ will be arbitrarily small for any $x_1, x_2 \in (-l, l)$ when the absolute value of the difference $|x_1 - x_2|$ is sufficiently small. These qualitative arguments show that the function $y = x^2$ is uniformly continuous on the interval $(-l, l)$.

Let us use now more exact arguments employing the definition of the uniform continuity. We specify an arbitrary $\varepsilon > 0$ and set $\delta = \varepsilon/(2l)$. Then $\forall x_1, x_2 \in (-l, l)$, which satisfy the inequality $|x_1 - x_2| < \delta$, there holds an inequality

$$|y(x_1) - y(x_2)| = |x_1 - x_2| \cdot |x_1 + x_2| < \delta \cdot 2l = \varepsilon.$$

And this means, according to the definition, that the function $y = x^2$ is uniformly continuous on the interval $(-l, l)$.

2nd technique. Let us consider the function $y = x^2$ on the closed interval $[-l, l]$. It is continuous on this interval and, consequently, according to Cantor's theorem, is uniformly continuous on it. It follows that the function $y = x^2$ is uniformly continuous on the interval $(-l, l)$. Indeed, $(-l, l) \subset [-l, l]$, and since the inequality $|y(x_1) - y(x_2)| < \varepsilon$ is satisfied $\forall x_1, x_2 \in [-l, l]$, which satisfy the inequality $|x_1 - x_2| < \delta(\varepsilon)$, it is satisfied for any $x_1, x_2 \in (-l, l)$ which satisfy the same inequality.

3rd technique. The derivative $y'(x) = 2x$ is bounded on the interval $(-l, l)$: $|y'(x)| = 2|x| \leqslant 2l$. According to Theorem 6, it follows from this that the function $y = x^2$ is uniformly continuous on $(-l, l)$. ▲

2. Test the function $y = x^2$ for the uniform continuity on the entire number line.

△ It can be seen from expression (1) that if $x_1, x_2 \in (-\infty, +\infty)$, then, for an arbitrarily small absolute value of the difference $|x_1 - x_2|$ the absolute value of the difference $|y(x_1) - y(x_2)|$ will not be small for sufficiently large x_1 and x_2 because of the factor $(x_1 + x_2)$. This qualitative reasoning implies that the function $y = x^2$ is not uniformly continuous throughout the number line $(-\infty, +\infty)$. We shall prove this by using the negation of the definition of the uniform continuity. We must prove that $\exists \varepsilon > 0$ such that $\forall \delta > 0$ $\exists x_1, x_2$, which satisfy the inequality $|x_1 - x_2| < \delta$, for which $|y(x_1) - y(x_2)| \geqslant \varepsilon$.

We take $\varepsilon = 1$ and $\forall \delta > 0$ set $x_1 = \frac{1}{\delta} + \frac{\delta}{2}$, $x_2 = \frac{1}{\delta}$. Then $|x_1 - x_2| = \delta/2 < \delta$, but then

$$|y(x_1) - y(x_2)| = |x_1 - x_2|\,|x_1 + x_2| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) = 1 + \frac{\delta^2}{4} \geqslant 1 = \varepsilon.$$

This proves that the function $y = x^2$ is not uniformly continuous on $(-\infty, +\infty)$. ▲

IV. Problems and Exercises for Independent Work

15. Using the definition of the uniform continuity and its negation, prove that the function $y = 1/x$: (a) is

uniformly continuous on the half-open interval $[1, +\infty)$, (b) is not uniformly continuous on the interval $(0, +\infty)$

16. Give an example of a function which is continuous on some open interval but is not uniformly continuous on it.

17. Prove the uniform continuity of the following functions using only the definition of the uniform continuity [i.e. using the specified arbitrary ε, we choose the necessary $\delta = \delta(\varepsilon)$]: (a) $f(x) = kx + b$ on $(-\infty, +\infty)$, $k \neq 0$, (b) $f(x) = x^3$ on $(-3, 5)$, (c) $f(x) = \sin x$ on $(-\infty, +\infty)$, (d) $f(x) = e$ on $[0, 10]$.

18. Use any technique to test each of the following functions for the uniform continuity: (a) $f(x) = \ln x$ on $(0, 1)$ and on $(1, 2)$, (b) $f(x) = \sin(1/x)$ on $(0, 1)$ and on $(0.01, 1)$, (c) $f(x) = \arctan x$ on $(-\infty, +\infty)$, (d) $f(x) = \arcsin x$ on $(-1, 1)$, (e) $f(x) = \sqrt{x}$ on $[0, +\infty)$.

19. Prove that the function $f(x) = \frac{|\sin x|}{x}$ is uniformly continuous on the intervals $I_1 = (-1 < x < 0)$ and $I_2 = (0 < x < 1)$ but is not uniformly continuous on their sum $I_1 + I_2 = \{0 < |x| < 1\}$.

20. Prove that if the function $f(x)$ is uniformly continuous on each of the closed intervals $[a, c]$ and $[c, b]$, then it is uniformly continuous on the closed interval $[a, b]$.

21. (a) Prove that if the function $f(x)$ is defined and continuous on the half-line $[a, +\infty)$ and there is a limit $\lim\limits_{x\to+\infty} f(x)$, then $f(x)$ is uniformly continuous on $[a, +\infty)$.

(b) Give an example of a function which is uniformly continuous on the half-line $[a, +\infty)$ and for which the limit $\lim\limits_{x\to+\infty} f(x)$ does not exist.

22. Give an example of a function which possesses an unbounded derivative on the set X but is uniformly continuous on that set.

23. Prove that a function which is uniformly continuous on an open interval is bounded on that interval. Is the converse statement true?

24. Prove that the sum and the product of two functions uniformly continuous on an open interval are uniformly continuous on that interval.

25. The modulus of continuity of the function $f(x)$ on

the interval (a, b) (a and b may be equal to $-\infty$ and $+\infty$ respectively) is the following function of the agrument δ ($\delta > 0$):

$$\omega_f(\delta) = \sup_{\substack{x_1, x_2 \in (a, b) \\ |x_1 - x_2| \leqslant \delta}} |f(x_1) - f(x_2)|$$

(if $|f(x_1) - f(x_2)|$ is an unbounded function for $\{|x_1 - x_2| \leqslant \delta, x_1, x_2 \in (a, b)\}$, then we write $\omega_f(\delta) = +\infty$).

(a) Prove that for the function $f(x)$ to be uniformly continuous on the interval (a, b), it is necessary and sufficient that $\lim\limits_{\delta \to 0} \omega_f(\delta) = 0$.

(b) Give an example of a function $f(x)$, $x \in (a, b)$, for which $\omega_f(\delta) = +\infty$.

26. Assume that the function $f(x)$ is continuous on the set X, i.e. continuous at every point $x \in X$. Then $\forall \varepsilon > 0$ and $\forall x \in X$ $\exists \delta = \delta(\varepsilon, x) > 0$ such that the inequality $|x' - x| < \delta(\varepsilon, x)$ ($x' \in X$) implies an inequality $|f(x') - f(x)| < \varepsilon$. Prove that for the function $f(x)$ to be uniformly continuous on the set X, it is necessary and sufficient that $\inf\limits_{X} \delta(\varepsilon, x) > 0$ ($\forall \varepsilon > 0$).

6.3. Some Theorems on Differentiable Functions

I. Fundamental Concepts and Theorems

1. Increase of a function at a point

Definition. *The function $f(x)$ is said to be* ***increasing at a point*** *x_0 if there is a neighbourhood of the point x_0 in which $f(x) > f(x_0)$ for $x > x_0$, $f(x) < f(x_0)$ for $x < x_0$.*

A decrease of a function at a point can be defined by analogy.

Theorem 7 (a sufficient condition for an increase of a function at a point). *If the function $f(x)$ is differentiable at a point x_0 and $f'(x_0) > 0$ ($f'(x_0) < 0$), then $f(x)$ increases (decreases) at the point x_0.*

2. Theorems on an increase and decrease of a function on an interval

Definition. *We say that the function $f(x)$* ***increases (not decreases)*** *on an interval X if $\forall x_1, x_2 \in X$ the condition*

$x_1 < x_2$ implies an inequality $f(x_1) < f(x_2)$ [$f(x_1) \leqslant f(x_2)$ respectively].

A decrease (nonincrease) of a function on an interval can be defined by analogy.

Theorem 8. *For the function $f(x)$ differentiable on an interval X not to decrease (not to increase) on that interval, it is necessary and sufficient that $\forall x \in X$ the inequality $f'(x) \geqslant 0$ ($f'(x) \leqslant 0$) be satisfied.*

Theorem 9 (a sufficient condition for strict monotonicity of a function). *If $f'(x) > 0$ ($f'(x) < 0$) $\forall x \in X$, then $f(x)$ increases (decreases) on the interval X.*

3. Theorems of Rolle, Lagrange and Cauchy

Theorem 10 (Rolle's theorem). *Let the function $f(x)$ satisfy the following conditions:*

(1^0) $f(x)$ is continuous on $[a, b]$,

(2^0) $f(x)$ is differentiable in (a, b),

(3^0) $f(a) = f(b)$.

Then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

Physical meaning of Rolle's theorem. Assume that x is time and $f(x)$ is the coordinate of a point, which moves along a straight line at the moment x. At the initial moment $x = a$ the point has a coordinate $f(a)$, then moves in a certain way with velocity $f'(x)$ and returns to the point with the coordinate $f(a)$ at the moment $x = b$ [$f(b) = f(a)$]. It is clear that to return to the point $f(a)$, it must stop at a certain moment (before "turning back"), i.e. at a certain moment $x = c$ the velocity $f'(c) = 0$.

Geometrical meaning of Rolle's theorem. There is a point $c \in (a, b)$ such that a tangent to the graph of the function $y = f(x)$ at the point $(c, f(c))$ is parallel to the x-axis.

Theorem 11 (Lagrange's theorem). *Let the function $f(x)$ satisfy the following conditions:*

(1^0) $f(x)$ is continuous on $[a, b]$,

(2^0) $f(x)$ is differentiable in (a, b).

Then there is a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a). \qquad (1)$$

Formula (1) is known as **Lagrange's formul** (or the *first mean value theorem*).

Physical meaning of Lagrange's theorem. Assume that x is time and $f(x)$ is the coordinate of a point, which moves along a straight line at the moment x. We write

Lagrange's formula in the form

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

The quantity on the left-hand side of the relation is, evidently, the average velocity of movement of the point along the straight line during the time interval from a to b. Lagrange's formula shows that there is a moment $x=c$ at which the instantaneous velocity is equal to the average velocity on the time interval $[a, b]$.

Geometrical meaning of Lagrange's theorem. The number $\frac{f(b)-f(a)}{b-a}$ is the slope of the straight line which passes through the endpoints of the graph of the function $y=f(x)$, i.e. through the points $(a, f(a))$ and $(b, f(b))$, and $f'(c)$ is the slope of the tangent to the graph at the point $(c, f(c))$. Lagrange's formula shows that the tangent to the graph at a point $(c, f(c))$ is parallel to the straight line which passes through the endpoints of the graph (or coincides with it).

Theorem 12 (Cauchy's theorem). *Let the functions $f(x)$ and $g(x)$ satisfy the following conditions:*

(1^0) *$f(x)$ and $g(x)$ are continuous on $[a, b]$,*

(2^0) *$f(x)$ and $g(x)$ are differentiable in (a, b),*

(3^0) *$g'(x) \neq 0$ $\forall x \in (a, b)$.*

Then there is a point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}. \tag{2}$$

Formula (2) is known as **Cauchy's formula.**

II. Control Questions and Assignments

1. Give a definition of an increase (decrease) of a function at a point.

2. Formulate the theorem on a sufficient condition for an increase of a function at a point.

3. Find out whether each of the following statements is true:

(a) "If a function increases at a point x_0, then it has a positive derivative at that point".

(b) "If the function $f(x)$ differentiable at a point x_0 increases at that point, then $f'(x_0) > 0$".

4. Formulate the theorem on a necessary and sufficient condition for the monotonicity of a differentiable function on an interval.

5. Formulate the theorem on a sufficient condition for the strict monotonicity of a differentiable function on an interval.

6. Find out whether the following statement is true: "If the function $f(x)$ differentiable on an interval X increases on that interval, then $f'(x) > 0 \quad \forall x \in X$".

7. Let the function $f(x)$ be defined in a neighbourhood of every point of the set X. Find out whether each of the following statements is true:

(a) "If $f(x)$ increases on the set X, then it increases at every point $x_0 \in X$".

(b) "If $f(x)$ increases at every point $x_0 \in X$, then it increases on the set X". (Consider the function $f(x) = -1/x$.)

8. Formulate Rolle's theorem.

9. Will Rolle's theorem remain true if we omit one of the following conditions: (a) $f(a) = f(b)$, (b) $f(x)$ is continuous on $[a, b]$? Give the corresponding examples.

10. Formulate Lagrange's theorem.

11. Formulate Cauchy's theorem.

III. Worked Problems

1. Find the intervals of monotonicity of the function $f(x) = 3x - x^3$.

△ We have $f'(x) = 3 - 3x^2 = 3(1 - x^2)$. Since $f'(x) > 0$ for $x \in (-1, 1)$, $f'(x) < 0$ for $x \in (-\infty, -1)$ and $x \in (1, \infty)$, it follows that the function $f(x) = 3x - x^3$ increases on the interval $(-1, 1)$ and decreases on the half-lines $(-\infty, -1)$ and $(1, +\infty)$; we can also state that $f(x)$ increases on the closed interval $[-1, 1]$ and decreases on the half-lines $(-\infty, -1]$ and $[1, +\infty)$. ▲

2. Prove that the function

$$f(x) = \begin{cases} x + x^2 \sin(2/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

increases at the point $x = 0$ but does not increase on any interval $(-\varepsilon, \varepsilon)$ ($\varepsilon > 0$ is an arbitrary number).

△ We have

$$f'(x) = \begin{cases} 1 + 2x\sin(2/x) - 2\cos(2/x) & \text{for } x \neq 0, \\ 1 & \text{for } x = 0 \end{cases}$$

(see Example 6 in 4.1 for the calculation of $f'(0)$).

Since $f'(0) = 1 > 0$, it follows, according to Theorem 7, that the function $f(x)$ increases at the point $x = 0$.

If the function $f(x)$ were increasing on an interval $(-\varepsilon, \varepsilon)$, then according to Theorem 8, the condition $f'(x) \geqslant 0$ would be satisfied $\forall x \in (-\varepsilon, \varepsilon)$. We shall show that this is not so. We set $x_n = 1/(\pi n)$ (n is a natural number). It is evident that $\forall \varepsilon > 0$ $\exists n$ such that $1/(\pi n) < \varepsilon$, i.e. $x_n \in (-\varepsilon, \varepsilon)$. Substituting $x = x_n = 1/(\pi n)$ into the expression for $f'(x)$ when $x \neq 0$, we get $f'(x_n) = -1 < 0$. This proves that the function $f(x)$ is not increasing on any interval $(-\varepsilon, \varepsilon)$. ▲

3. Let the function $f(x)$ satisfy the following conditions: (1) $f(x)$ possesses a continuous derivative on $[a, b]$, (2) $f(x)$ possesses a second derivative in (a, b), (3) $f(a) = f'(a) = 0$, $f(b) = 0$. Prove that there is a point $c \in (a, b)$ such that $f''(c) = 0$.

△ All conditions of Rolle's theorem are evidently satisfied for the function $f(x)$ on the closed interval $[a, b]$. Therefore there is a point $d \in (a, b)$ such that $f'(d) = 0$.

We consider the function $f'(x)$ on the interval $[a, d]$. We have (1) $f'(x)$ is continuous on $[a, d]$, (2) $f'(x)$ possesses a derivative $(f'(x))' = f''(x)$ in (a, d), (3) $f'(a) = f'(d) = 0$. By virtue of Rolle's theorem, there is a point $c \in (a, d)$ [and, consequently, $c \in (a, b)$] such that $(f'(x))' \mid_{x=c} = f''(c) = 0$. ▲

4. Prove that $|\cos x - \cos y| \leqslant |x - y|$ $\forall x, y$.

△ From Lagrange's formula we have

$$\cos x - \cos y = \sin \xi \cdot (x - y),$$

where ξ is a point from the interval (x, y). Since $|\sin \xi| \leqslant 1$, it follows that $|\cos x - \cos y| \leqslant |x - y|$. ▲

5. Assume that the functions $f(x)$ and $g(x)$ are defined and differentiable for $x \geqslant x_0$, and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$. Prove that $f(x) > g(x)$ for $x > x_0$.

△ Let us consider a function $\varphi(x) = f(x) - g(x)$ on an arbitrary closed interval $[x_0, x]$ $(x > x_0)$. From

Lagrange's formula we have

$$\varphi(x) - \varphi(x_0) = \varphi'(\xi)(x - x_0), \quad (3)$$

where ξ is a point from the interval (x_0, x). Since

$$\varphi(x_0) = f(x_0) - g(x_0) = 0,$$

$$\varphi'(\xi) = f'(\xi) - g'(\xi) > 0, \quad x - x_0 > 0,$$

we find, from relation (3), that $\varphi(x) > 0$, i.e. $f(x) - g(x) > 0$ for $x > x_0$. Thus $f(x) > g(x)$ for $x > x_0$. ▲

IV. Problems and Exercises for Independent Work

27. Find the intervals of monotonicity of the following functions:

(a) $f(x) = ax^2 + bx + c \ (a > 0)$, (b) $f(x) = x^3 + 3x^2 + 3x$,

(c) $f(x) = \dfrac{2x}{1+x^2}$, (d) $f(x) = x + \sin x$,

(e) $f(x) = x + 2\sin x$, (f) $f(x) = \sin(\pi/x)$,

(g) $f(x) = x^2 2^{-x}$, (h) $f(x) = x^n e^{-x} \ (n > 0, x \geqslant 0)$.

28. Prove that if a function increases at every point of an open interval, then it increases on that interval. Will the statement remain true if we replace the interval by an arbitrary set?

29. Let the function $f(x)$ satisfy the following conditions: (1) $f(x)$ has a continuous $(n-1)$th derivative on $[x_0, x_n]$, (2) $f(x)$ has an nth derivative in (x_0, x_n), (3) $f(x_0) = f(x_1) = \ldots = f(x_n)$, where $x_0 < x_1 \ldots < x_n$. Prove that there is a point $\xi \in (x_0, x_n)$ such that $f^{(n)}(\xi) = 0$.

30. Using Rolle's theorem, prove that if all the roots of the polynomial

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n \quad (a_0 \neq 0)$$

with real coefficients a_k $(k = 0, 1, \ldots, n)$ are real, then its derivatives $P_n'(x)$, $P_n''(x)$, $\ldots$, $P_n^{(n-1)}(x)$ also have only real roots.

31. Prove that all the roots of Legendre's polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \{(x^2 - 1)^n\}$$

are real and lie in the interval $(-1, 1)$.

32. Find a point c in the first mean value formula (1) for the function

$$f(x)=\begin{cases} 0.5\,(3-x^2) & \text{for } 0\leqslant x\leqslant 1,\\ 1/x & \text{for } 1<x<+\infty \end{cases}$$

on the interval [0, 2].

33. Using Lagrange's formula, prove the validity of the following inequalities:

(a) $|\sin x-\sin y|\leqslant|x-y|\quad \forall x,\ y,$

(b) $|\arctan x-\arctan y|\leqslant|x-y|\quad \forall x,\ y,$

(c) $\frac{x-y}{x}<\ln\frac{x}{y}<\frac{x-y}{y}$ for $0<y<x.$

34. Prove the validity of the following inequalities:

(a) $e^x>1+x$ for $x\neq 0,$

(b) $x-\frac{x^2}{2}<\ln(1+x)<x$ for $x>0,$

(c) $\ln(1+x)>\frac{x}{1+x}$ for $x>0,$

(d) $x-\frac{x^3}{3!}<\sin x<x$ for $x>0,$

(e) $x+\frac{x^3}{3}\tan x$ for $0<x<\pi/2.$

Give a geometric illustration of these inequalities.

35. Assume that the functions $f(x)$ and $g(x)$ are defined and n times differentiable for $x\geqslant x_0$, and $f^{(k)}(x_0)=g^{(k)}(x_0)$ $(k=0,\ 1,\ \ldots,\ n-1)$; $f^{(n)}(x)>g^{(n)}(x)$ for $x>x_0$. Prove that $f(x)>g(x)$ for $x>x_0$.

36. Prove that if a function is differentiable but unbounded on an open interval, then its derivative is also unbounded on that interval. Give an example showing that the converse statement is not true.

37. Is Cauchy's formula valid for the functions $f(x)=x^2$ and $g(x)=x^3$ on the interval $[-1,\ 1]$? What condition of Cauchy's theorem is not satisfied for these functions?

38. Let the function $f(x)$ satisfy the following conditions: (1) $f(x)$ is continuous on $[a,\ b]$, (2) $f(x)$ is differentiable in $(a,\ b)$, (3) $f(x)$ is not a linear function. Prove that there is a point $c\in(a,\ b)$ such that $|f(b)-f(a)|<|f'(c)|\cdot|b-a|$.

39. Let the function $f(x)$ satisfy the following conditions: (1) $f(x)$ is twice differentiable on $[a,\ b]$; (2) $f'(a)=f'(b)=0$. Prove that there is a point $c\in(a,\ b)$ such that $|f(b)-f(a)|\leqslant(1/4)(b-a)^2|f''(c)|$.

40. During the time t s the point covered the distance of s m moving along a straight line. At the initial and the final moment the velocity of the point is zero. Prove that at some moment the absolute value of the acceleration of the point was not smaller than $4s/t^2$ m/s^2.

6.4. L'Hospital's Rule

I. Fundamental Concepts and Theorems

Theorem 13. *Assume that the following conditions are fulfilled:*

(1^0) *the functions $f(x)$ and $g(x)$ are defined and differentiable in a neighbourhood of the point a (except, maybe, the point a itself),*

(2^0) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$,

(3^0) $g'(x) \neq 0$ *in that neighbourhood of the point a (except, maybe, the point a itself),*

(4^0) *there is limit* $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

Then there is a limit $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ *and it is equal to* $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

Remark. If all conditions of Theorem 13 are fulfilled in the right-hand (left-hand) half-neighbourhood of the point a, then the theorem holds true for the right-hand (left-hand) limit of the function $f(x)/g(x)$ at the point a.

Theorem 14. *Assume that the following conditions are fulfilled:*

(1^0) *the functions $f(x)$ and $g(x)$ are defined and differentiable on the half-line $(a, +\infty)$,*

(2^0) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} g(x) = 0$,

(3^0) $g'(x) \neq 0 \quad \forall x \in (a, +\infty)$,

(4^0) *there is a limit* $\lim\limits_{x\to+\infty} \dfrac{f'(x)}{g'(x)}$.

Then there is a limit $\lim\limits_{x\to+\infty} \dfrac{f(x)}{g(x)}$ *and is equal to* $\lim\limits_{x\to+\infty} \dfrac{f'(x)}{g'(x)}$.

Remark. If we replace condition 4^0 in Theorems 13 and 14 by the condition $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \infty$ (a is a number or a symbol $+\infty$) then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \infty$.

Theorems 13 and 14 make it possible to find the values of indeterminate forms of type 0/0.

Theorem 15. *If conditions* 1^0, 3^0 *and* 4^0 *of Theorems* 13 *and* 14 *are fulfilled and the condition* $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \infty$ (a is a number or a symbol $+\infty$) *is fulfilled instead of condition* 2^0, *then there is a limit* $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ *and it is equal to* $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

Theorem 15 makes it possible to find the value of indeterminate forms of type ∞/∞. It is also valid in the case of one-sided limits.

Each of the Theorems 13-15 is called **L'Hospital's rule.**

Indeterminate forms of other types ($0\cdot\infty$, $\infty - \infty$, 1^∞, 0^0, ∞^0) can be reduced to indeterminate forms of type $0/0$ or ∞/∞ and then we can apply L'Hospital's rule.

II. Control Questions and Assignments

1. Formulate L'Hospital's rule of finding the values of indeterminate forms of type (a) 0/0 as $x \to a$, (b) 0/0 as $x \to +\infty$, (c) ∞/∞ as $x \to a$, (d) ∞/∞ as $x \to +\infty$.

2. Assume that conditions 1^0-3^0 of Theorem 13 (or Theorem 14, or Theorem 15) are fulfilled and there is no limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$. Does it follow that the limit $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ does not exist? Consider the following examples:

$$\text{(a)} \lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x}, \quad \text{(b)} \lim_{x\to +\infty} \frac{x+\sin x}{2x+\sin x}.$$

III. Worked Problems

1. Find $\lim\limits_{x\to 0} \frac{\sin \alpha x}{\tan \beta x}$.

$\triangle$ This limit is an indeterminate form of type 0/0. Let us verify whether the conditions of Theorem 13 are fulfilled:

(1^0) the functions $\sin \alpha x$ and $\tan \beta x$ are defined and differentiable in a neighbourhood of the point $x = 0$,

(2^0) $\lim\limits_{x\to 0} \sin \alpha x = \lim\limits_{x\to 0} \tan \beta x = 0$,

(3^0) $(\tan \beta x)' = \dfrac{\beta}{\cos^2 \beta x} \neq 0$ in the neighbourhood of the point $x = 0$,

(4^0) $\lim\limits_{x\to 0} \dfrac{(\sin \alpha x)'}{(\tan \beta x)'} = \lim\limits_{x\to 0} \dfrac{\alpha \cos \alpha x}{\dfrac{\beta}{\cos^2 \beta x}} = \dfrac{\alpha}{\beta}$.

Consequently, according to Theorem 13 $\lim\limits_{x\to 0} \dfrac{\sin \alpha x}{\tan \beta x} = \dfrac{\alpha}{\beta}$. ▲

Sometimes, to find the value of an indeterminate form, we have to apply L'Hospital's rule several times in succession, as in the following example:

2. Find

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}. \tag{1}$$

△ This limit is an indeterminate form of type 0/0. Conditions 1^0-3^0 of Theorem 13 are fulfilled and the limit of the ratio of derivatives

$$\lim_{x\to 0} \frac{(\tan x - x)'}{(x^3)'} = \lim_{x\to 0} \frac{\dfrac{1}{\cos^2 x} - 1}{3x^2} \tag{2}$$

is also an indeterminate form of type 0/0.

Conditions 1^0-3^0 of Theorem 13 are fulfilled for limit (2) and the limit of the ratio of derivatives

$$\lim_{x\to 0} \frac{\left(\dfrac{1}{\cos^2 x} - 1\right)'}{(3x^2)'} = \lim_{x\to 0} \frac{2\cos^{-3} x \sin x}{6x} \tag{3}$$

is again an indeterminate form of type 0/0.

To find the value of this indeterminate form, we can also use L'Hospital's rule since conditions 1^0-3^0 of Theorem 13 are fulfilled for limit (3) and the limit of the ratio of derivatives

$$\lim_{x\to 0} \frac{(2\cos^{-3} x \sin x)'}{(6x')} = \lim_{x\to 0} \frac{6\cos^{-4} x \sin^2 x + 2\cos^{-2} x}{6} = \frac{1}{3}. \tag{4}$$

Thus, by virtue of (2)-(4), the required limit (1) is equal to 1/3. ▲

3. Find $\lim\limits_{x\to+0} x^x$.

△ This limit is an indeterminate form of type 0^0. We represent x^x in the form $e^{x\ln x}$ and consider $\lim\limits_{x\to+0}(x\ln x)$. This limit is an indeterminate form $0\cdot\infty$. Writing $x\ln x$ as $\dfrac{\ln x}{1/x}$, we arrive at an indeterminate form of type ∞/∞. It is easy to verify that all conditions of Theorem 15 for one-sided limits are fulfilled for $\lim\limits_{x\to+0}\dfrac{\ln x}{1/x}$ (verify this independently). Applying L'Hospital's rule, we obtain

$$\lim_{x\to+0}\frac{\ln x}{1/x}=\lim_{x\to+0}\frac{1/x}{-1/x^2}=\lim_{x\to+0}(-x)=0.$$

It follows that $\lim\limits_{x\to+0} x^x=\lim\limits_{x\to+0} e^{x\ln x}=e^0=1$. ▲

IV. Problems and Exercises for Independent Work

Find each of the following limits:

41. $\lim\limits_{x\to+\infty}\dfrac{\ln x}{x^\alpha}$ $(\alpha>0)$. 42. $\lim\limits_{x\to+\infty}\dfrac{x^\alpha}{a^x}$ $(\alpha>0,\ a>1)$.

43. $\lim\limits_{x\to0}\dfrac{\sin\alpha x}{\sin\beta x}$. 44. $\lim\limits_{x\to0}\dfrac{\tan x-x}{\sin x-x}$.

45. $\lim\limits_{x\to0}\dfrac{\cosh x-\cos x}{x^2}$.

46. $\lim\limits_{x\to\pi/2}\left(x-\dfrac{\pi}{2}\right)\cot 2x$. 47. $\lim\limits_{x\to3\pi/4}\dfrac{1+\sqrt[3]{\tan x}}{1-2\cos^2 x}$.

48. $\lim\limits_{x\to0}\dfrac{\arcsin x}{x}$. 49. $\lim\limits_{x\to0}\dfrac{x}{\arccos x}$.

50. $\lim\limits_{x\to0}\dfrac{\ln(\sin\alpha x)}{\ln(\sin\beta x)}$ $(\alpha>0,\beta>0)$.

51. $\lim\limits_{x\to0}\dfrac{\cos(\sin x)-\cos x}{x^4}$. 52. $\lim\limits_{x\to+0}\dfrac{e^{-1/x}}{x^\alpha}$ $(\alpha>0)$.

53. $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x-a}$. 54. $\lim\limits_{x\to1}(2-x)^{\tan(\pi x/2)}$.

55. $\lim\limits_{x\to\pi/4}(\tan x)^{\tan 2x}$. 56. $\lim\limits_{x\to+\infty}x^{1/x}$.

57. $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$. 58. $\lim\limits_{x\to0}\left(\dfrac{\arctan x}{x}\right)^{1/x^2}$.

59. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$. 60. $\lim\limits_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$.

61. $\lim\limits_{x\to 0}\left(\cot x-\frac{1}{x}\right)$. 62. $\lim\limits_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$.

63. $\lim\limits_{x\to a}\frac{x^x-a^x}{x-a}$. 64. $\lim\limits_{x\to 1-0}(\arccos x)^{1-x}$.

6.5. Taylor's Formula

I. Fundamental Concepts and Theorems

1. Taylor's polynomial. Let the function $f(x)$ be n times differentiable at a point x_0. The polynomial

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is *Taylor's polynomial* for the function $f(x)$ (with centre at the point x_0). It possesses the following properties:

$$P_n^{(k)}(x_0)=f^{(k)}(x_0)\quad (k=0,\,1,\,\ldots,\,n).$$

The following theorem discloses the part played by Taylor's polynomial.

Theorem 16. *If the function $f(x)$ is defined in a neighbourhood of a point x_0 and n times differentiable at the point x_0, then*

$$f(x)=P_n(x)=R_{n+1}(x), \tag{1}$$

where $R_{n+1}(x)=o\,((x-x_0)^n)$.

Formula (1) is known as **Taylor's formula** for the function $f(x)$ with centre at the point x_0 and the *remainder* $R_{n+1}(x)$ in *Peano's form.*

2. Various forms of the remainder

Theorem 17. *Let the function $f(x)$ be defined and $n+1$ times differentiable in a neighbourhood of the point x_0. Assume that x is an arbitrary value of the argument belonging to that neighbourhood and $p>0$ is an arbitrary number. Then there is a point $\xi\in(x_0,\,x)$ such that*

$$R_{n+1}(x)=f(x)-P_n(x)=\frac{(x-\xi)^{n+1}}{n!\,p}\left(\frac{x-x_0}{x-\xi}\right)^p f^{(n+1)}(\xi). \tag{2}$$

Expression (2) is known as the *general form of the remainder.*

The following special cases of the general form of a remainder are especially important:

(a) *Lagrange's form* ($p = n+1$):

$$R_{n+1}(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(x_0 + \theta(x-x_0)) \quad (0<\theta<1),$$

(b) *Cauchy's form* ($p=1$):

$$R_{n+1}(x) = \frac{(x-x_0)^{n+1}(1-\theta)^n}{n!} f^{(n+1)}(x_0+\theta(x-x_0))$$

$$(0<\theta<1).$$

3. **Fundamental expansions.** If $x_0 = 0$, Taylor's formula is usually called **Maclaurin's formula.** The most significant expansions by Maclaurin's formula are:

I. $e^x = \sum\limits_{k=0}^{n} \frac{x^k}{k!} + R_{n+1}(x).$

II. $\sin x = \sum\limits_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + R_{2n+1}(x).$

III. $\cos x = \sum\limits_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + R_{2n+2}(x).$

IV. $\ln(1+x) = \sum\limits_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + R_{n+1}(x).$

V. $(1+x)^\alpha = 1 + \sum\limits_{k=1}^{n} \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} x^k + R_{n+1}(x).$

II. Control Questions and Assignments

1. What is Taylor's polynomial for the function $f(x)$ with centre at the point x_0? What property does it possess?
2. Formulate the theorem on Taylor's formula with the remainder (a) in Peano's form, (b) in the general form. How do the conditions of these theorems differ? The conditions of which theorem follow from the other one?
3. Use the general form of the remainder to derive Lagrange's and Cauchy's forms. Obtain Peano's form of the remainder from Lagrange's form.

4. Write Maclaurin's formula for the function $f(x)$, and the remainder of this formula in the forms of Peano, Lagrange and Cauchy.

5. Write the fundamental expansions and the remainders of those expansions in the forms of Peano, Lagrange and Cauchy.

III. Worked Problems

1. Expand the function $\tan x$ in Maclaurin's series up to the term with x^3 inclusive.

△ We seek the derivatives of the function $f(x) = \tan x$ up to the third order inclusive:

$f'(x) = \dfrac{1}{\cos^2 x} = \cos^{-2} x;$

$f''(x) = 2\cos^{-3} x \sin x;$

$f'''(x) = 6\cos^{-4} x \sin^2 x + 2\cos^{-2} x.$

From this we find that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 2$. From Maclaurin's formula with the remainder in Peano's form we have

$$\tan x = x + \frac{x^3}{3} + o(x^3).$$

Note that the calculation of $f^{(4)}(x)$ yields $f^{(4)}(0) = 0$. Therefore we can write the remainder as $o(x^4)$. ▲

2. Expand the function $f(x) = \ln \cos x$ in Maclaurin's series up to the term with x^4 inclusive.

△It is not necessary here to calculate the derivatives of $f(x)$ up to the fourth order inclusive, instead we can use the fundamental expansions III and IV. Using expansion III, we obtain

$$\ln(\cos x) = \ln\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) = \ln(1 + t),$$

where $t = -\dfrac{x^2}{2} + \dfrac{x^4}{24} + o(x^4)$.

We shall now make use of expansion IV:

$$\ln\cos x = \ln(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2$$

$$+ o\left(\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2\right)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4)$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + o(x^4). \blacktriangle$$

3. Estimate the absolute error of the approximate formula

$$e^x \cong 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} = P_n(x) \tag{3}$$

$$\text{for } 0 \leqslant x \leqslant 1.$$

△ To estimate the absolute error, we must evaluate the remainder $R_{n+1}(x) = e^x - P_n(x)$. The remainder $R_{n+1}(x)$ in Lagrange's form for the function e^x has the form $R_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} e^{\theta x}$ $(0 < \theta < 1)$. From this we find that

$$| R_{n+1}(x) | \leqslant \frac{e}{(n+1)!} \quad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{4}$$

This is the required estimation of the absolute error of the approximate formula (3) for $0 \leqslant x \leqslant 1$. ▲

4. Solve the following problem using estimate (4): how many terms must we take in formula (1) for $x = 1$ to calculate the number e with an accuracy to within 10^{-6}?

It is easy to calculate that $10! > 3 \cdot 10^6$. Therefore $\frac{e}{10!} < \frac{3}{3 \cdot 10^6} = 10^{-6}$. Thus it is sufficient to set $n = 9$ in formula (3) for $x = 1$ to get the number e with an accuracy to within 10^{-6}. ▲

5. Using the fundamental expansions, find

$$\lim_{x \to 0} \frac{\tan x + 2 \sin x - 3x}{x^4}.$$

△ We have

$$\lim_{x \to 0} \frac{\tan x + 2 \sin x - 3x}{x^4}$$

$$= \lim_{x \to 0} \frac{x + \frac{x^3}{3} + o(x^4) + 2\left(x - \frac{x^3}{6} + o(x^4)\right) - 3x}{x^4}$$

$$= \lim_{x \to 0} \frac{o(x^4)}{x^4} = 0. \blacktriangle$$

IV. Problems and Exercises for Independent Work

65. Use Maclaurin's formula to expand the function (x) up to the term of the indicated order inclusive:
(a) $f(x) = e^{-x}$ up to the term with x^n,
(b) $f(x) = e^{2x-x^2}$ up to the term with x^5,
(c) $f(x) = \sin\sin x$ up to the term with x^3,
(d) $f(x) = \cos\sin x$ up to the term with x^4,
(e) $f(x) = \ln\frac{\sin x}{x}$ up to the term with x^6,
(f) $f(x) = \sqrt[3]{\sin x^3}$ up to the term with x^{13},
(g) $f(x) = \sqrt[n]{a^n + x}$ up to the term with x^2 $(a > 0)$,
(h) $f(x) = \sqrt{1 - x + 2x^2}$ up to the term with x^3.

66. Write the expansion of each of the following functions in Taylor's series with centre at the point $x = 1$:
(a) $f(x) = x^2$, (b) $f(x) = \sqrt{x}$ up to the term with $(x - 1)^3$, (c) $f(x) = \sin(\pi x/2)$ up to the term with $(x - 1)^4$.

67. Estimate the absolute error of the following approximate formulas:
(a) $\sin \cong x - \frac{x^3}{6}$ for $|x| \leqslant \frac{1}{2}$,
(b) $\tan x \cong x + \frac{x^3}{3}$ for $|x| \leqslant 0.1$,
(c) $\sqrt{1+x} \cong 1 + \frac{x}{2} - \frac{x^2}{8}$ for $0 \leqslant x \leqslant 1$.

68. Use Taylor's formula to find the following approximate values:
(a) $\sqrt[3]{9}$ with an accuracy to within 10^{-3},
(b) $\sqrt{90}$ with an accuracy to within 10^{-4},
(c) $\sin 18°$ with an accuracy to within 10^{-4}.
(d) $\sin 1°$ with an accuracy to within 10^{-8},
(e) $\ln 1.1$ with an accuracy to within 10^{-3},
(f) $e^{0.2}$ with an accuracy to within 10^{-5},
(g) $\cos 6°$ with an accuracy to within 10^{-5}.

69. Using the fundamental expansions, find the following limits:
(a) $\lim\limits_{x\to 0} \frac{\cos x - e^{-x^2/2}}{x^4}$, (b) $\lim\limits_{x\to 0} \frac{\sin 2x - 2\tan x}{\ln(1+x^3)}$,
(c) $\lim\limits_{x\to +\infty} \sqrt{e^x}\left(\sqrt{e^x + 1} - \sqrt{e^x - 1}\right)$,

(d) $\lim\limits_{x\to+\infty} x^{3/2}(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x})$,

(e) $\lim\limits_{x\to 0} \frac{e^x+e^{-x}-2}{2x^2}$; (f) $\lim\limits_{x\to 0} \left(\frac{1}{x}-\frac{1}{\sin x}\right)$,

(g) $\lim\limits_{x\to 0} \frac{1}{x}\left(\frac{1}{x}-\cot x\right)$, (h) $\lim\limits_{x\to 0} \frac{\sin \sin x - x\sqrt[3]{1-x^2}}{x^5}$.

Chapter 7

Investigating the Behaviour of a Function and Constructing Graphs

7.1. Constructing the Graphs of Explicit Functions

I. Fundamental Concepts and Theorems

1. Asymptotes to the graph of a function. A function defined by the relation $y = f(x)$, $x \in D(f)$, is called an *explicit* function.

Definition. *The straight line $x = c$ is a **vertical asymptote** to the graph of the function $y = f(x)$ if at least one of the limits $\lim\limits_{x\to c-0} f(x)$ or $\lim\limits_{x\to c+0} f(x)$ is equal to $+\infty$ or $-\infty$.*

Definition. *The straight line $y = kx + b$ is an **oblique asymptote** to the graph of the function $y = f(x)$ as $x \to +\infty$ if this function can be represented as $f(x) = kx + b + \alpha(x)$, where $\alpha(x) \to 0$ as $x \to +\infty$.*

Theorem 1. *For the straight line $y = kx + b$ to be an oblique asymptote to the graph of the function $y = f(x)$ as $x \to +\infty$, it is necessary and sufficient that the limits*

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k, \qquad \lim_{x\to+\infty} [f(x) - kx] = b$$

should exist.

By analogy we can introduce the concept of an oblique asymptote to the graph of a function as $x \to -\infty$.

2. Even, odd and periodic functions

Definition. *The function $y = f(x)$ is **even** if*

$$\forall x \in D(f):\ f(x) = f(-x).$$

Definition. *The function $y = f(x)$ is* ***odd*** *if*

$$\forall x \in D(f):\quad f(x) = -f(-x).$$

Definition. *The function $y = f(x)$ is* ***periodic****, if there is a number $T \neq 0$, called the* ***period*** *of the function $y = f(x)$, such that*

$$\forall x \in D(f): f(x) = f(x + T) = f(x - T).$$

By the period of a function we usually mean the least of the positive periods, provided that it exists.

3. Local extremum of a function. Assume that the function $y = f(x)$ is defined in a neighbourhood of a point x_0.

Definition. *The function $y = f(x)$ is said to have a* ***local maximum (minimum)*** *at the point x_0 if there is a neighbourhood of the point x_0 in which the inequality $f(x) < f(x_0)$ [$f(x) > f(x_0)$* respectively] *is satisfied for $x \neq x_0$.* A local maximum and a local minimum are united under the general term ***local extremum*** (or simply ***extremum***).

Theorem 2 (a necessary condition for an extremum). *If the function $y = f(x)$ has an extremum at the point x_0, then the derivative $f'(x)$ at the point x_0 is either zero or does not exist.*

The values of the argument of the function $y = f(x)$ for which the derivative is either equal to zero or does not exist but the function itself is continuous are known as *points of a possible extremum.*

Theorem 3 (the first sufficient condition for an extremum). *Assume that the function $y = f(x)$ is differentiable in a neighbourhood of a point x_0 of a possible extremum (except, maybe, for the point x_0 itself). Then, if the derivative $f'(x)$ changes sign from plus to minus (from minus to plus) when passing through the point x_0 (in the direction of the increase of x), then the function $y = f(x)$ has a local maximum (minimum) at the point x_0. If the derivative of the function does not change sign when passing through the point x_0, then the function $y = f(x)$ does not possess an extremum at the point x_0.*

Theorem 4 (the second sufficient condition for an extremum). *Assume that the function $y = f(x)$ has the second derivative at the point x_0 of a possible extremum. Then, if $f''(x_0) < 0$ $(f''(x_0) > 0)$, then the function $y = f(x)$ possesses local maximum (minimum) at the point x_0.*

4. The direction of convexity and the points of inflection of the graph of a function. Assume that the function $y = f(x)$ has a finite derivative at every point of the interval (a, b). Then, at every point $M(x, f(x))$, $x \in (a, b)$, the graph of the function $y = f(x)$ has a tangent which is nonparallel to the y-axis.

Definition. *The graph of a function is said to be* ***convex downwards (upwards)*** *on the interval (a, b) if, within the interval (a, b), the graph lies not lower (not higher) than any tangent.*

Theorem 5. *If the function $y = f(x)$ has a second derivative on the interval (a, b) and if that derivative is nonnegative (nonpositive) everywhere on this interval, then the graph of the function $y = f(x)$ has convexity directed downwards (upwards) on the interval (a, b).*

Definition. *The point $M(c, f(c))$ of the graph of the function $y = f(x)$ is a* ***point of inflection*** *of this graph if at this point the graph has a tangent and there is a neighbourhood of the point c within which on the left and on the right of the point c the directions of convexity of the graph of the function $y = f(x)$ are different.* We also say that at the point $M(c, f(c))$ the graph of the function has an *inflection.*

Theorem 6 (a necessary condition for inflection). *If the graph of the function $y = f(x)$ has an inflection at the point $M(c, f(c))$ and the second derivative $f''(x)$ is continuous at the point c, then $f''(c) = 0$.*

Theorem 7 (a sufficient condition for inflection). *If the second derivative of the function $y = f(x)$ exists in a neighbourhood of the point c, with $f''(c) = 0$, and the signs of $f''(x)$ are different on the left and on the right of the point c, then the graph of the function has an inflection at the point $M(c, f(c))$.*

5. Scheme of constructing the graph of the function $y = f(x)$.

1°. Find the domain of definition of a function and the values of that function at the points of discontinuity and the boundary points of the domain.

If the function suffers a discontinuity at the point c and $f(c + 0)$ or $f(c - 0)$ vanishes, then $x = c$ is a vertical asymptote to the graph of the function $y = f(x)$.

If the function is defined on a half-line or on the whole number line, then we must find out (using Theorem 1) whether the graph of the function has oblique asymptotes.

If oblique asymptotes do not exist, then we must find out whether the function is bounded as $x \to \infty$ or unbounded (in the last case we must find out whether it is infinitely large as $x \to \infty$ and what sign it has).

2°. Find out whether the function is even, odd or periodic.

The aim of this item is to make the calculations as short as possible. Indeed, if the function is even or odd, then we may consider the part of the domain which belongs to the positive semi-axis of abscissas rather than the whole domain. On this part of the domain we must carry out complete investigation of the behaviour of the function and construct its graph, then, resorting to symmetry, complete the construction on the whole of the domain.

If the function is periodic, then it is sufficient to investigate the behaviour of the function on any closed interval whose length is equal to the period of the function and then, constructing the graph on that interval, extend it to the whole of the domain of the function.

3°. Find the zeros of the function, i.e. solve the equation $f(x) = 0$. These solutions and the points of discontinuity of the function divide its domain of definition into intervals where the function is of constant sign.

4°. Find local extrema and the intervals of increase and decrease of the function (we shall depict the extremal points on the graph by circles ○).

5°. Find the intervals where the graph of the function retains the direction of its convexity and the points of inflection (we shall depict the points of inflection on the graph by crosses: $\times$ or $+$).

II. Control Questions and Assignments

1. Give a definition and an example of a vertical asymptote to the graph of a function.

2. Give a definition and cite an example of an oblique asymptote to the graph of a function as $x \to +\infty$ (as $x \to -\infty$).

3. Formulate the theorem which defines necessary and sufficient conditions for the existence of an oblique asymptote to the graph of a function.

4. Give examples of a function which has oblique asymptotes to its graph as $x \to +\infty$ and as $x \to -\infty$ and those asymptotes (a) coincide, (b) do not coincide.

5. Give a definition of a local extremum of a function.

6. What is a point of a possible extremum of a function?

7. Formulate the theorem which defines a necessary condition for an extremum (a) of an arbitrary function, (b) of a differentiable function. Give an example showing that this condition is not sufficient.

8. Formulate the theorems which express the sufficient conditions for an extremum of a function.

9. Give a definition of the direction of convexity of the graph of a function.

10. Give a definition of a point of inflection of the graph of a function.

11. Can the direction of convexity of the graph of a function change when the function passes through the point which is not a point of inflection? Give examples.

12. Formulate the necessary condition for inflection of the graph of a function. Give an example showing that this condition is not sufficient.

13. Formulate the sufficient condition for inflection of the graph of a function.

14. Present a scheme for constructing the graph of the function $y = f(x)$.

III. Worked Problems

1. Construct the graph of the function $y = \arcsin \frac{2x}{1+x^2}$.

$\triangle$ 1°. The function is defined for the values of x for which, as follows from the definition of the arc sine, the inequality $\left|\frac{2x}{1+x^2}\right| \leqslant 1$ is satisfied. It is equivalent to the inequality $(1-|x|)^2 \geqslant 0$. The last inequality is valid for any real x. Thus $D(f) = \mathbf{R}$. The function $\frac{2x}{1+x^2}$ is continuous at any point (being the quotient of the division of two continuous functions). Therefore the function $\arcsin \frac{2x}{1+x^2}$ is also continuous at any point (being the superposition of continuous functions) and, consequently, there are no vertical asymptotes

to the graph of the function. To find the vertical asymptote as $x \to +\infty$, we calculate the following limits:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1}{x} \arcsin \frac{2x}{1+x^2} = 0,$$

$$\lim_{x\to+\infty} [f(x) - kx] = \lim_{x\to+\infty} \arcsin \frac{2x}{1+x^2} = \arcsin 0 = 0.$$

Hence it follows that the straight line $y = 0$ is an asymptote as $x \to +\infty$ (it is rather horizontal than oblique). We can establish by analogy that the same straight line $y = 0$ is an asymptote as $x \to -\infty$.

2°. The function is evidently nonperiodic and odd and therefore it is sufficient to consider the half-line $[0, +\infty)$ rather than the whole domain.

3°. We have $y = 0$ for $x = 0$. There are no other zeros or points of discontinuity of the function. On the half-line $(0, +\infty)$ the function is positive.

4°. We seek the points of a possible extremum on the half-line $[0, +\infty)$ for which purpose we calculate the derivative of the function $x \neq 1$:

$$y' = \frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}} \frac{2(1+x^2)-4x^2}{(1+x^2)^2}$$

$$= \frac{1+x^2}{|1-x^2|} \cdot \frac{2(1-x^2)}{(1+x^2)^2} = \frac{2 \operatorname{sgn}(1-x^2)}{1+x^2}.$$

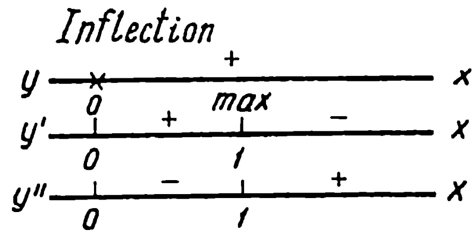
It can be seen that the derivative does not vanish at any point. Since $y'(1+0) = -1$, $y'(1-0) = 1$, the derivative does not exist at the point $x = 1$. When passing through the point $x = 1$, the derivative changes sign from plus to minus. Therefore, at the point $x = 1$, the function has a local maximum and $y(1) = \arcsin 1 = \pi/2$. Note that at the point $x = 1$ the function is continuous and its derivative has a discontinuity of the first kind. In that case the corresponding point on the graph of the function [the point $(1, \pi/2)$ in this example] is known as a *corner point*. The intervals of monotonicity of the function are defined by the sign of the derivative: $y' > 0$ for $0 \leqslant x < 1$ and $y' < 0$ for $x > 1$.

5°. Since the second derivative

$$y'' = \frac{-4x \operatorname{sgn}(1-x^2)}{(1+x^2)^2}, \quad x \neq 1,$$

vanishes only for $x = 0$ and y'' changes sign when passing through the point $x = 0$, the graph of the function has an inflection at the point $(0, y(0)) = (0, 0)$. The direction of convexity is defined by the sign of the second derivative: $y'' < 0$ for $0 \leqslant x < 1$, $y'' > 0$ for $x > 1$.

We have completed the investigation of the behaviour of the function. Before constructing the graph, it is convenient to depict the results of the investigation on the scheme, in particular, the intervals where the function, the first derivative y' and the second derivative y'' are of constant sign:



Reading now the information on the scheme, we construct the graph of the function on the interval $[0, +\infty)$. On the closed interval $[0, 1]$ (a) the function increases from the

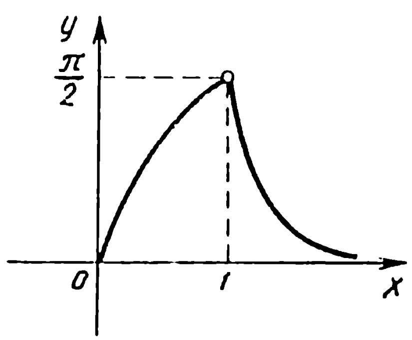


Fig. 6

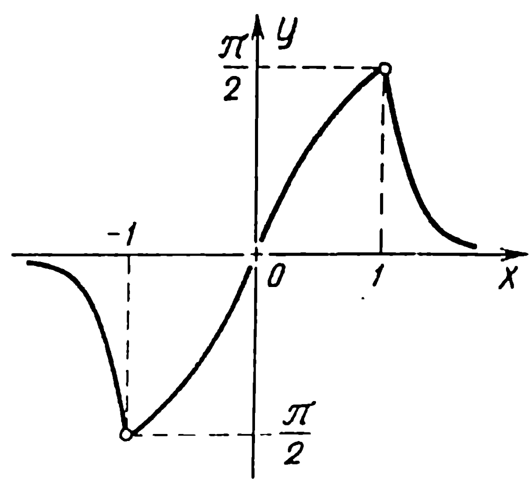


Fig. 7

value $y = 0$ for $x = 0$ to the value $y = \pi/2$ for $x = 1$, (b) the graph is convex upwards. Next, on the half-line $[1, +\infty)$ (a) the function decreases remaining positive, (b) the graph is convex downwards, (c) as $x \to +\infty$, the graph approaches the asymptote, the x-axis. Note that when passing through the point $x = 1$, the direction of convexity of the graph changes but the point $(1, \pi/2)$ is not an inflection point, it is a corner point (Fig. 6).

Finally, using the fact that the function is odd, we complete the construction of its graph on the whole of the domain of definition (Fig. 7). ▲

Remark. If a curve is defined by an equation $\Phi(x, y) = 0$ and if this equation can be solved for y or for x, then the construction of the curve reduces to the construction of the graphs of explicit functions.

2. Construct a curve defined by the equation $y^2 - \sin^4 x = 0$.

△ $\forall x \in \mathbf{R}$ this equation has two solutions with respect to y: $y = \sin^2 x$ and $y = -\sin^2 x$ which are explicit functions defined throughout the number line. The graphs of these functions are symmetric about the x-axis. It is therefore sufficient to construct the graph of the first function and then, resorting to symmetry, construct the whole curve. Thus the problem has reduced to the construction of the graph of the explicit function $y = \sin^2 x$ which we shall write as

$$y = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

We use the scheme presented above to investigate the behaviour of this function.

1°. We have $D(y) = \mathbf{R}$.

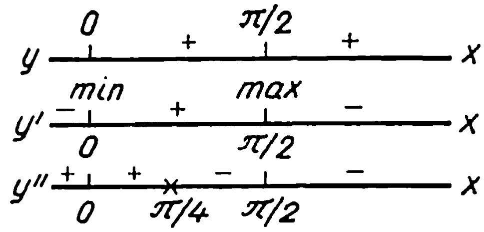
2°. The function $y(x)$ is periodic with period $T = \pi$. Therefore, to construct the graph of the function, it is sufficient to consider an intercept on the x-axis of length π, say, $[-\pi/2, \pi/2]$. Since, in addition, $y(x)$ is even, we can restrict our consideration to the closed interval $[0, \pi/2]$.

3°. We seek the zeros of the function on the closed interval $[0, \pi/2]$; we have $\frac{1}{2} - \frac{1}{2}\cos 2x = 0$ for $x = k\pi$, $k \in \mathbf{Z}$, but of all these solutions only $x = 0$ belongs to the interval $[0, \pi/2]$. The function has no points of discontinuity. On the interval $(0, \pi/2]$ the function is positive.

4°. We seek $y' = \sin 2x$. On the closed interval $[0, \pi/2]$ the derivative is equal to zero for $x = 0$ and $x = \pi/2$. Furthermore, $y' > 0$ for $0 < x < \pi/2$, $y' < 0$ for $x < 0$ and $x > \pi/2$. According to Theorem 3, the function has a local maximum at the point $\pi/2$, with $y(\pi/2) = 1$, and a local minimum at the point $x = 0$, with $y(0) = 0$.

The whole interval $[0, \pi/2]$ is an interval of increase of the function.

5⁰. We have $y'' = 2\cos 2x$. On the interval $[0, \pi/2]$ the second derivative vanishes for $x = \pi/4$. When passing through this point, y'' changes sign. This means, according to Theorem 7, that the graph of the function has an inflection at the point $(\pi/4, y(\pi/4)) = (\pi/4, 1/2)$. Thus we can construct the following scheme:



Reading the information on the scheme, we construct the graph of the function on the closed interval $[0, \pi/2]$ (Fig. 8). Using the fact that the function is even, we complete the construction of its graph on the interval $[-\pi/2, \pi/2]$ (Fig. 9).

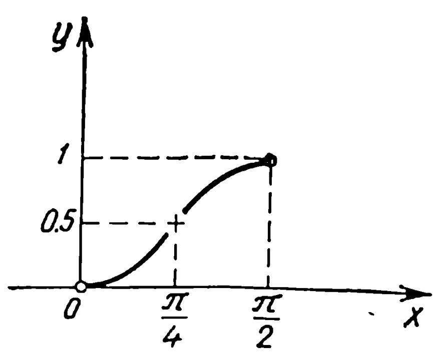


Fig. 8

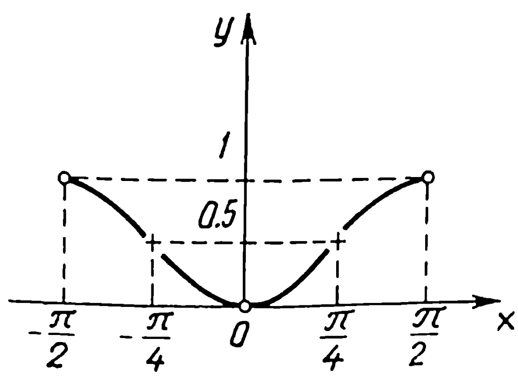


Fig. 9

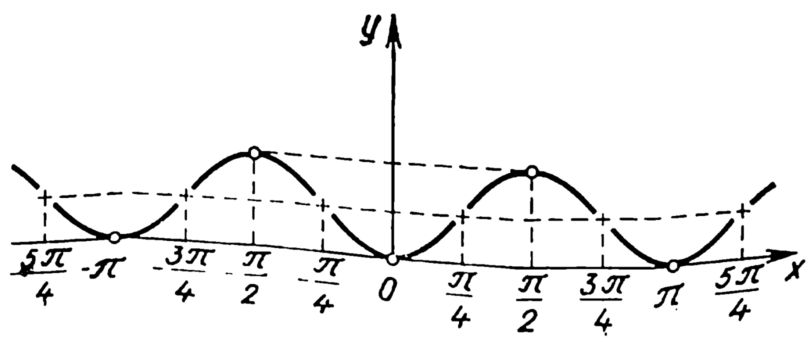


Fig. 10

Bearing in mind that the function is periodic, we construct its graph on the whole domain of definition (Fig. 10).

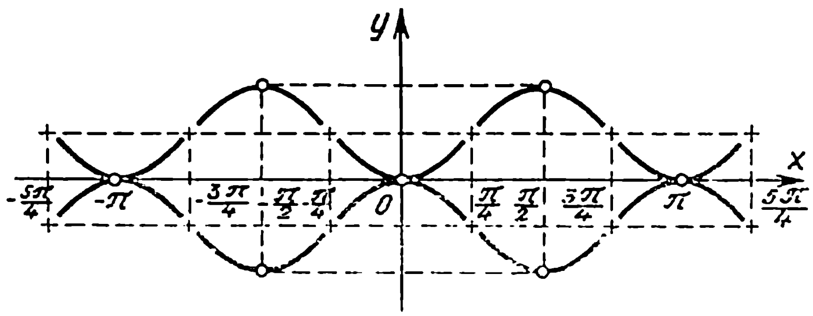


Fig. 11

Finally, using the symmetry of the original curve about the x-axis, we obtain the whole curve (Fig. 11). ▲

IV. Problems and Exercises for Independent Work

Construct the graph of each of the following explicit functions:

1. $y = 1 + x^2 - 0.5x^4$. 2. $y = (x+1)(x-2)^2$.
3. $y = 0.4x - 0.5x^3 + 0.1x^5$. 4. $y = (1-x^2)^{-1}$.
5. $y = x^4(1+x)^{-3}$. 6. $y = (1+x)^4(1-x)^{-4}$.
7. $y = x^2(x-1)(x+1)^{-2}$. 8. $y = x(1-x^2)^{-2}$.
9. $y = 2x - 1 + (x+1)^{-1}$. 10. $y = \dfrac{\cos x}{\cos 2x}$.
11. $y = \arccos\dfrac{1-x^2}{1+x^2}$. 12. $y = \arcsin(\sin x)$.
13. $y = \sin(\arcsin x)$. 14. $y = \arctan(\tan x)$.
15. $y = \arctan(1/x)$. 16. $y = (x+2)e^{1/x}$.
17. $y = 0.5(\sqrt{x^2+x+1} - \sqrt{x^2-x+1})$.
18. $y = \sqrt{x^2+1} - \sqrt{x^2-1}$.
19. $y = (x+2)^{2/3} - (x-2)^{2/3}$.
20. $y = (x+1)^{2/3} + (x-1)^{2/3}$.

Construct the curves defined by the following equations:

21. $y^2 = 8x^2 - x^4$. 22. $y^2 = (x-1)(x-2)(x-3)$.
23. $y^2 = (x-1)(x+1)^{-1}$.
24. $y^2 = x^2(1-x)(1+x)^{-2}$.
25. $y^2 = x^4(x+1)$.
26. $x^2(y-2)^2 + 2xy - y^2 = 0$.

7.2. Investigation of Plane Curves Represented Parametrically

I. Scheme for Investigation of a Curve

The parametric equations of a plane curve have the form

$$x = x(t), \quad y = y(t), \quad t \in T. \tag{1}$$

We can use the following scheme to investigate and construct such a curve.

1°. We find the set T which is the common part of the domains of definition of the functions $x(t)$ and $y(t)$ (if the set T is not specified), noting, in particular, the values of the parameter t_i (including $t_i = \pm\infty$) for which at least one of the one-sided limits $\lim\limits_{t \to t_i \pm 0} x(t)$, $\lim\limits_{t \to t_i \pm 0} y(t)$ is equal to $+\infty$ or $-\infty$.

2°. We find out whether the curve possesses symmetry which makes it possible to cut down the calculations.

3°. We find the zeros of the functions $x(t)$ and $y(t)$ and the domains where the functions are of constant sign.

4°. We find the points t_k at which at least one of the derivatives $\dot{x}(t)$, $\dot{y}(t)$ is zero or discontinuous. Note that the points t_i, indicated in 1°, and the points t_k, found in this item, divide the set T into intervals where the derivatives $\dot{x}(t)$ and $\dot{y}(t)$ are of constant sign. Therefore, on each of these intervals (t_p, t_{p+1}) the function $x(t)$ is strictly monotonic and, consequently, on the interval (t_p, t_{p+1}) the system of equations (1) gives parametrically a function of the form $y = f(x)$ (see 4.1). The derivatives of this function are expressed by the formulas

$$f' = \frac{\dot{y}(t)}{\dot{x}(t)}, \quad f'' = \frac{\frac{\mathrm{d}}{\mathrm{d}t}(f')}{\dot{x}(t)}.$$

We shall call the part of the curve corresponding to the change of the parameter t from t_p to t_{p+1} a *branch* of the curve. Every branch of the curve is the graph of a function of the form $y = f(x)$.

5°. We find the points t_j at which $f'' = 0$.

6°. We compile a table of the form

(t_p, t_{p+1})		...	
(x_p, x_{p+1})		...	
(y_p, y_{p+1})		...	
Sign of f''		...	

In the first row we write the intervals of variation of the parameter t whose boundary points t_p and t_{p+1} are the points we found in items 1°, 4° and 5°. In the second and third rows of the table we present the corresponding intervals of variation of the variables x and y. In the last row we give the sign of f'' which defines the direction of convexity of the graph of the corresponding branch of the curve.

7°. Using the table, we construct the branches of the curve corresponding to the intervals (t_p, t_{p+1}).

Remark 1. In 1° of the scheme we can find the asymptotes to the curve (provided that they exist). To do that, we must bear in mind the following:

(a) if $x \to x_0$ and $y \to \infty$ as $t \to t_p$ ($t \to t_p + 0$ or $t \to t_p - 0$), then $x = x_0$ is a vertical asymptote to the curve,

(b) if $x \to \infty$ and $y \to y_0$ as $t \to t_p$ ($t \to t_p + 0$ or $t \to t_p - 0$), then $y = y_0$ is a horizontal asymptote to the curve,

(c) if $x \to \infty$ and $y \to \infty$ as $t \to t_p$ ($t \to t_p + 0$ or $t \to t_p - 0$), then an oblique asymptote is possible which can be found in accordance with Theorem 1.

Remark 2. When studying the symmetry of a curve (item 2° of the scheme), four cases must be borne in mind when it suffices to consider only the nonnegative part of the domain T rather than the whole domain:

(a) $\forall t \in T$: $x(t) = x(-t)$, $y(t) = -y(-t)$ (symmetry about the x-axis),

(b) $\forall t \in T$: $x(t) = -x(-t)$, $y(t) = y(-t)$ (symmetry about the y-axis),

(c) $\forall t \in T$: $x(t) = -x(-t)$, $y(t) = -y(-t)$ (symmetry about the origin),

(d) $\forall t \in T$: $x(t) = x(-t)$, $y(t) = y(-t)$ (superposition).

Remark 3. If t_p is a point found in item 4° of the scheme and if $\dot{x}(t)$ retains sign on the interval (t_{p-1}, t_{p+1}), then, on this interval, the system of equations (1) defines parametrically a function of the form $y = f(x)$ for which the point $x(t_p)$ is the point of a possible extremum. We can find out whether $x(t_p)$ is the point of extremum of the function $y = f(x)$ by considering the variation of y on the intervals (t_{p-1}, t_p) and (t_p, t_{p+1}).

Remark 4. When investigating a curve, we may encounter one of the specific singular points of the curve defined parametrically, a cuspidal point or cusp (see Example 2 in III).

II. Control Questions and Assignments

1. How can we calculate the derivatives of a function represented in parametric form?
2. A curve is represented in parametric form: $x = \sin^2 t$, $y = \cos^2 t$. What interval is it sufficient to consider for the point $(x(t), y(t))$ to appear at every point of the curve only once when the parameter t varies on that interval?
3. How can we find the asymptotes to a curve represented in parametric form?
4. How can the symmetry of a curve represented in parametric form be investigated and used?
5. Formulate the necessary condition for a local extremum of a function represented in parametric form.
6. Give a scheme of investigation and construction of a curve represented in parametric form.

III. Worked Problems

1. Construct the following curve represented in parametric form:

$$x = \frac{t}{1-t^2}, \qquad y = \frac{t(1-2t^2)}{1-t^2}.$$

△ 1°. We have

$$t \in (-\infty, -1) \cup (-1, 1) \cup (1, +\infty),$$
$$x \in (0, +\infty) \cup (-\infty, +\infty) \cup (-\infty, 0),$$
$$y \in (-\infty, -\infty) \cup (+\infty, -\infty) \cup (+\infty, +\infty).$$

It follows that $x = 0$ is a vertical asymptote to the curve, and as $t \to -1$ and $t \to 1$ oblique asymptotes are possible. Indeed,

$$\lim_{x\to\pm\infty} \frac{y}{x} = \lim_{t\to 1\pm 0} (1-2t^2) = -1,$$

$$\lim_{x\to\pm\infty} (y+x) = \lim_{t\to 1\pm 0} 2t = 2.$$

By analogy we find the limits as $t \to -1$:

$$\lim_{x\to\pm\infty} \frac{y}{x} = -1; \quad \lim_{x\to\pm\infty} (y+x) = -2.$$

Thus the curve has two asymptotes $y = -x + 2$ and $y = -x - 2$.

2⁰. Since $x(t) = -x(-t)$, $y(t) = -y(-t)$, the curve is symmetric about the point $O(0, 0)$. It is therefore sufficient to consider the set $M = [0, 1) \cup (1, +\infty)$.

3⁰. On the set M we have $x(t) = 0$ for $t = 0$, $y(t) = 0$ for $t = 0$ and $t = 1/\sqrt{2}$.

4⁰. $\dot{x}(t) = \frac{1+t^2}{(1-t^2)^2}$, $\dot{y}(t) = \frac{2t^4-5t^2+1}{(1-t^2)^2}$. On the set M we have $\dot{x}(t) > 0$, and $\dot{y}(t) = 0$ for $t_1 = 0.5\sqrt{5-\sqrt{17}} \cong 0.47$ and $t_2 = 0.5\sqrt{5+\sqrt{17}} \cong 1.51$.

5⁰. $f' = \frac{\dot{y}}{\dot{x}} = \frac{2t^4 - 5t^2 + 1}{1+t^2}$, $f'' = \frac{\frac{d}{dt}(f')}{\dot{x}} = \frac{-4t(1-t^2)^3(3+t^2)}{(1+t^2)^3}$. Hence $f'' \leqslant 0$ for $t \in [0, 1)$, $f'' > 0$ for $t \in (1, +\infty)$.

6⁰. We compile a table:

(t_p, t_{p+1})	$(0, 0.47)$	$(0.47, 1)$	$(1, 1.51)$	$(1.51, +\infty)$
(x_p, x_{p+1})	$(0, +\infty)$	$(0.6, +\infty)$	$(-\infty, -0.7)$	$(-0.7, 0)$
(y_p, y_{p+1})	$(0, 0.3)$	$(0.3, -\infty)$	$(+\infty, 2.3)$	$(2.3, +\infty)$
Sign of f''	+	+	−	−

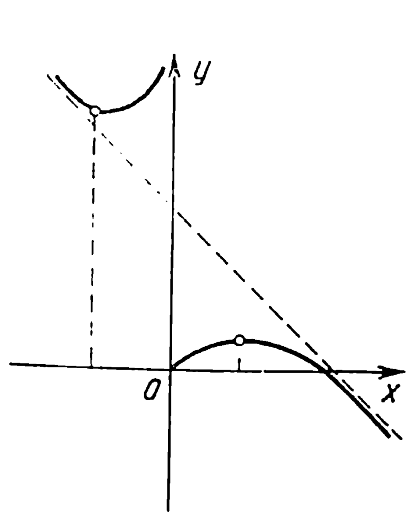


Fig. 12

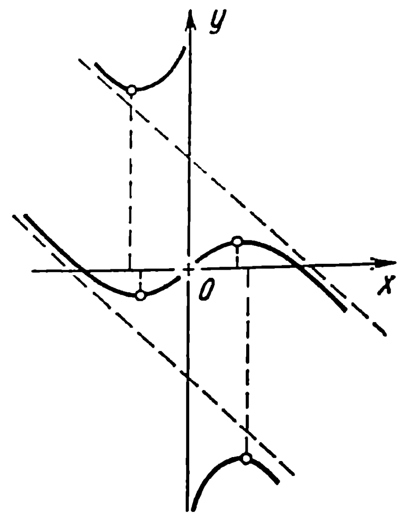


Fig. 13

7⁰. We construct the part of the curve corresponding to the set M (Fig. 12). Next, resorting to the symmetry of the curve, we construct the whole curve (Fig. 13). ▲

2. We construct a curve represented in parametric form:

$$x = 2t - t^2, \quad y = 3t - t^3. \tag{2}$$

△ 1⁰. We have

$$t \in (-\infty, +\infty),$$
$$x \in (-\infty, -\infty),$$
$$y \in (+\infty, -\infty).$$

Thus, as $x \to -\infty$ $(t \to \pm\infty)$, oblique asymptotes are possible. However, $\lim\limits_{x\to-\infty} \frac{y}{x} = \lim\limits_{t\to\pm\infty} \frac{3t-t^3}{2t-t^2} = \infty$, i.e. there are no asymptotes.

2⁰. The curve does not possess the properties of symmetry and periodicity.

3⁰. We have $x = 0$ for $t = 0$ and $t = 2$; $y = 0$ for $t = 0$, $t = -\sqrt{3}$ and $t = \sqrt{3}$.

4⁰. We find that $\dot{x}(t) = 2(1 - t) = 0$ for $t = 1$, $\dot{y}(t) = 3(1 - t^2) = 0$ for $t = -1$ and $t = 1$.

5⁰. Since $f'' = \frac{3}{4(1-t)}$, it follows that $f'' > 0$ for $t < 1$, $f'' < 0$ for $t > 1$.

6°. We compile a table:

$(t_p,\ t_{p+1})$	$(-\infty, -1)$	$(-1, 1)$	$(1, +\infty)$
$(x_p,\ x_{p+1})$	$(-\infty, -3)$	$(-3, 1)$	$(1, -\infty)$
$(y_p,\ y_{p+1})$	$(+\infty, -2)$	$(-2, 2)$	$(2, -\infty)$
Sign of f''	$+$	$+$	$-$

7°. We construct a curve (Fig. 14).

Note that if we consider t to be time and the curve defined by the system of equations (2) to be the trajectory of motion of the point on the xy-plane, then

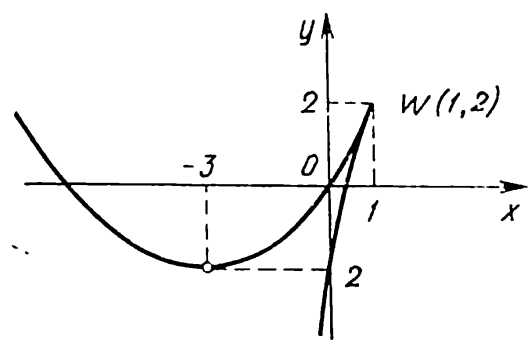


Fig. 14

$\{\dot{x}, \dot{y}\}$ is the velocity vector of the motion of that point. For $t = 1$ in this example we have $\dot{x}(t) = \dot{y}(t) = 0$, i.e. the velocity is zero and $\dot{x}(t)$ and $\dot{y}(t)$ change sign when passing through $t = 1$. This means that as $t \to 1-0$ the point moving along the trajectory approaches the point $W(1, 2)$ (see Fig. 14), stops at the point W at the moment $t = 1$ and then moves in the opposite direction. Since $\lim\limits_{t\to 1-0} \frac{\dot{y}(t)}{\dot{x}(t)} = \lim\limits_{t\to 1+0} \frac{\dot{y}(t)}{\dot{x}(t)}$, the branches of the trajectory corresponding to $t \leqslant 1$ and $t \geqslant 1$ have the same one-sided tangent for $t = 1$, i.e. at the point $W(1, 2)$. The point $W(1, 2)$ is known as a *cuspidal point*, or *cusp* (this name evidently corresponds to the physical interpretation considered above). ▲

Remark 1. It is sometimes possible to obtain parametric equations for the curve defined by an equation of the form

$$F(x, y) = 0. \tag{3}$$

Here is the way to do this. We set $y = \alpha(t)\, x^n$, where $\alpha(t)$ and n are the requisite function and number. Substituting the expression for y into equation (3), we get $F(x, \alpha(t)\, x^n) = 0$. Let $x = \varphi(t)$ be a solution of this equation. Then

$$x = \varphi(t), \quad y = \alpha(t)\, \varphi^n(t) \equiv \psi(t)$$

is a parametric equation of this curve. In practical applications the choice of the function $\alpha(t)$ is determined by the form of the function $F(x, y)$.

Let us consider a curve defined by the equation

$$x^4 + y^4 = 2xy. \tag{4}$$

This equation can be satisfied by the coordinates x and y of only those points which lie in quadrants I and III or on the coordinate axes, i.e. the inequality $xy \geqslant 0$ must be satisfied. To pass to the parametric equations of the curve, we set $y = x\sqrt{\tan t}$. Substituting this expression into equation (4), we obtain

$$x^4(1 + \tan^2 t) = 2x^2\sqrt{\tan t},$$

whence $x = 0$ and $x = \sqrt[4]{4\tan t}\cos t$. The first solution $x = 0$ is contained in the second for $t = 0$. Thus the parametric equations of the curve have the form

$$x = \sqrt[4]{4\tan t}\cos t, \quad y = \sqrt[4]{4(\tan t)^3}\cos t.$$

However, we can introduce the parameter t in another way, setting, for instance, $y = xt$. Then we arrive at the following parametric equations of the curve:

$$x = \sqrt{\frac{2t}{1+t^4}}, \quad y = \sqrt{\frac{2t^3}{1+t^4}}$$

and 1248

$$x = -\sqrt{\frac{2t}{1+t^4}}, \quad y = -\sqrt{\frac{2t^3}{1+t^4}}.$$

Investigate the curve for both cases of introduction of the parameter t as an exercise.

Remark 2. We can investigate a curve represented in polar coordinates using the scheme presented in this section. Indeed, assume that in the polar system of coordinates (φ, ρ) the curve is defined by the equation $\rho = f(\varphi)$. Then, expressing the Cartesian coordinates in terms of the polar ones

$$\begin{cases} x = \rho \cos \varphi, \\ y = \rho \sin \varphi, \end{cases}$$

we get the parametric equations of the curve (φ is a parameter)

$$x = f(\varphi) \cos \varphi, \quad y = f(\varphi) \sin \varphi.$$

IV. Problems and Exercises for Independent Work

Construct the curves defined by the following equations:

27. $x = \frac{1}{4}(t+1)^2, \ y = \frac{1}{4}(t-1)^2.$

28. $x = \frac{t^2}{1-t^2}, \quad y = \frac{1}{1+t^2}.$

29. $x = \frac{t^2}{t-1}, \quad y = \frac{t}{t^2-1}.$

30. $x = -5t^2 + 2t^5, \quad y = -3t^2 + 2t^3.$

31. $x = \frac{t^2+1}{4(1-t)}, \quad y = \frac{t}{t+1}.$

32. $x = \frac{(t+2)^2}{t+1}, \quad y = \frac{(t-2)^2}{t-1}.$

33. $x = \frac{t-t^2}{1+t^2}, \quad y = \frac{t^2-t^3}{1+t^2}.$

Passing to parametric equations, construct the curves defined by the following equations:

34. $x^3 + y^3 = 3axy$, where $a > 0$ (Cartesian folium).

35. $(x - a)^2 (x^2 + y^2) = b^2x^2$, where $a > 0$, $b > 0$ (conchoid of Nicomedes). Consider the following cases: (a) $a > b$, (b) $a = b$, (c) $a < b$; in each case define the character of the singular point of the curve $O(0, 0)$.

36. $x^{2/3} + y^{2/3} = a^{2/3}$, where $a > 0$ (astroid).

37. $x^6 + 2x^3y = y^3$. ● Set $y = x^2t$.

38. $4y^2 = 4x^2y + x^5$. ● Set $y = x^2t$.

39. $x^4 + 2y^3 = 4x^2y$. 40. $x^3 - 2x^2y - y^2 = 0$.

41. $x^2y^2 + y = 1$. ● Set $y = t/x$.

42. $x^3 + y^3 = 3x^2$. 43. $y^5 + x^4 = xy^2$.
44. $x^4 - y^4 + xy = 0$. 45. $x^5 + y^5 = xy^2$.
Construct the curves defined by the following equations in the polar system:
46. $\rho = 5/\varphi$ $(0 < \varphi < +\infty)$.
47. $\rho^2 = 2a^2 \cos^2 \varphi$.
48. $\rho = a \cos \varphi + b$.
49. $\rho = a \sin 3\varphi$ $(a > 0)$.
50. $\rho = 2/\sqrt{\cos 3\varphi}$.

Chapter 8

The Definite Integral

8.1. Integrability of a Function (in the Sense of Riemann) and the Definite Integral

I. Fundamental Concepts and Theorems

1. Integral sums and the definite integral. Let the function $f(x)$ be defined on the closed interval $[a, b]$ (where $a < b$). We shall designate the arbitrary partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \ldots < x_n = b$ into n subintervals $[x_{i-1}, x_i]$ $(i = 1, 2, 3, \ldots, n)$ as $T[a, b]$, or simply as T. We set $\Delta x_i = x_i - x_{i-1}$. On each subinterval $[x_{i-1}, x_i]$ we choose an arbitrary point ξ_i and form a sum:

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = I(x_i, \xi_i).$$

The number $I(x_i, \xi_i)$ is the *integral sum* of the function $f(x)$ which corresponds to this partition $T[a, b]$ and to this choice of the intermediate points ξ_i on the subintervals $[x_{i-1}, x_i]$. We introduce the designation $\Delta = \max\limits_{1 \leqslant i \leqslant n} \Delta x_i$.

Definition. *The number I is the* ***limit of the integral sums*** *$I(x_i, \xi_i)$ as $\Delta \to 0$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that for every partition $T[a, b]$, for which $\Delta < \delta$, the inequality $|I(x_i, \xi_i) - I| < \varepsilon$ holds true for any choice of the intermediate points ξ_i on $[x_{i-1}, x_i]$.*

Definition. *The function $f(x)$ is* ***Riemann integrable*** *on the closed interval $[a, b]$ if there is a limit $\lim\limits_{\Delta\to 0} I(x_i, \xi_i) = I$.*

In that case the number I is the *definite integral* of the function $f(x)$ over the interval $[a, b]$ and is written as

$$I = \int_a^b f(x)\, dx.$$

2. Darboux sums. Let $f(x)$ be defined and bounded on $[a, b]$. To carry out the arbitrary partition $T[a, b]$, we introduce the designations $m_i = \inf\limits_{[x_{i-1}, x_i]} f(x)$, $M_i = \sup\limits_{[x_{i-1}, x_i]} f(x)$ and form sums

$$s = \sum_{i=1}^{n} m_i \Delta x_i, \quad S = \sum_{i=1}^{n} M_i \Delta x_i.$$

The numbers s and S are the *lower* and the *upper sum* (*Darboux sums*) corresponding to this partition $T[a, b]$.

It is evident that $s \leqslant I(x_i, \xi_i) \leqslant S$ for a fixed partition $T[a, b]$ and any choice of intermediate points on this partition.

Here are the properties of Darboux sums.

1°. For any fixed partition

$$s = \inf_{\substack{\text{over all} \\ \text{collections of} \\ \text{points } \xi_i}} \{I(x_i, \xi_i)\}, \quad S = \sup_{\substack{\text{over all} \\ \text{collections of} \\ \text{points } \xi_i}} \{I(x_i, \xi_i)\}.$$

2°. If we have obtained partition T_2 from partition T_1 by adding several new points (i.e. by refining T_1), then the lower sum s_2 of the partition T_2 is not smaller than the lower sum s_1 of the partition T_1 and the upper sum S_2 of the partition T_2 is not larger than the upper sum S_1 of the partition T_1: $s_1 \leqslant s_2$, $S_2 \leqslant S_1$.

3°. The lower sum of an arbitrary partition does not exceed the upper sum of any other partition.

4°. Let $\{s\}$ and $\{S\}$ be the sets of various lower and upper sums for any partitions of $[a, b]$. The numbers

$$I = \inf_{\substack{\text{over} \\ \text{all partitions}}} \{S\}, \quad I = \sup_{\substack{\text{over} \\ \text{all partitions}}} \{s\}$$

are the *upper* and the *lower Darboux integral*, respectively.

The lower Darboux integral does not exceed the upper one: $\underline{I} \leqslant \overline{I}$.

5⁰. **Darboux lemma**

$$\lim_{\Delta \to 0} S = \overline{I}, \quad \lim_{\Delta \to 0} s = \underline{I}.$$

3. The necessary and sufficient conditions for integrability

Theorem 1. *For the function $f(x)$ bounded on the closed interval $[a, b]$ to be integrable on this interval, it is necessary and sufficient that $\underline{I} = \overline{I}$.*

Theorem 2. *For a function bounded on the closed interval $[a, b]$ to be integrable on this interval, it is necessary and sufficient that $\forall \varepsilon > 0$ there should be a partition T $[a, b]$ (at least one) for which*

$$S - s < \varepsilon. \tag{1}$$

Recall that the number $\omega_i = M_i - m_i$ is the oscillation of the function on the interval $[x_{i-1}, x_i]$.

We can write condition (1) in the form

$$S - s = \sum_{i=1}^{n} \omega_i \Delta x_i < \varepsilon.$$

4. Some classes of integrable functions

Theorem 3. *A function $f(x)$ continuous on the closed interval $[a, b]$ is integrable on this interval.*

Corollary. *Every elementary function is integrable on any closed interval which lies entirely in the domain of definition of that function (since it is continuous on this interval).*

Theorem 4. *Let $f(x)$ be bounded on the closed interval $[a, b]$. If $\forall \varepsilon > 0$ there is a finite number of intervals which cover all points of discontinuity of $f(x)$ and the sum of whose lengths is smaller than ε, then $f(x)$ is integrable on the interval $[a, b]$.*

Corollary. *A piecewise-continuous function (a function which has a finite number of points of discontinuity of the first kind on the closed interval $[a, b]$) is integrable on this interval.*

Remark. If the conditions of Theorem 4 are fulfilled, then the value of the integral $\int_a^b f(x)\,dx$ does not depend on the values of $f(x)$ at the points of discontinuity. We therefore often raise and solve the problem of calculation of the integral of a function which is not defined either at a finite number of points of the interval $[a, b]$ or on the set of points which can be covered by a finite number of intervals of an arbitrarily small length. In that case we assume that the definition of the function $f(x)$ is completed arbitrarily at these points but the function remains bounded on the interval $[a, b]$ and, consequently, integrable.

For example, strictly speaking, the integral

$$\int_0^1 \frac{\sin x}{x}\,dx \tag{2}$$

does not exist since at the point $x=0$ the function $\frac{\sin x}{x}$ is not defined. However, the integral $\int_0^1 f(x)\,dx$, where

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{for } x\neq 0,\\ C & \text{for } x=0\end{cases} \quad (C \text{ is an arbitrary number}),$$

exists and is independent of the choice of C. We therefore assume that integral (2) also exists and is equal to $\int_0^1 f(x)\,dx$.

Theorem 5. *A function $f(x)$ monotonic on the closed interval $[a, b]$ is integrable on this interval.*

II. Control Questions and Assignments

1. What is a partition of the closed interval $[a, b]$?
2. What is the integral sum of the function $f(x)$ on the closed interval $[a, b]$?
3. Give a definition of the limit of integral sums when the mesh size of the partition $(\Delta \to 0)$ approaches zero on the closed interval $[a, b]$.
4. What is the definite integral?

5. What is an integrable function?

6. Prove that an unbounded function is not integrable.

7. Is the function $f(x) = 1/x$ integrable on the closed interval [1, 2]? on the closed interval [—1, 1]?

8. Is the function $f(x) = \tan x \cot x$ integrable on the closed interval $[\pi/6, \pi/4]$? on the closed interval [—1, 1]?

9. Is the function $f(x) = e^{-1/x}$ integrable on the closed interval [—3, —2]? on the closed interval [—1, 0]? on the closed interval [—1, 1]?

10. Is every bounded function integrable? Substantiate the answer by giving examples.

11. What are the lower and upper Darboux sums?

12. Name the properties of the Darboux sums.

13. Formulate the necessary and sufficient conditions for integrability (two variants).

14. Recall the classes of integrable functions you know. Give examples of functions belonging to those classes.

15. Think of an example of a function monotonic on the closed interval $[a, b]$ which has an infinite number of points of discontinuity. Is such a function integrable on $[a, b]$?

III. Worked Problems

1. A constant function $f(x) = C$ is integrable on $[a, b]$ since the integral sums have the same value for any partition and any choice of the points ξ_i:

$$I(x_i, \xi_i) = \sum_{i=1}^{n} f(\xi_i)\, \Delta\xi_i = C \sum_{i=1}^{n} \Delta x_i = C(b-a).$$

Hence $\int_a^b C\, dx = \lim_{\Delta\to 0} I(x_i, \xi_i) = C(b-a)$.

2. Prove that the Dirichlet function

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

is not integrable on any closed interval $[a, b]$.

△ Indeed, on any arbitrarily small closed interval $[x_{i-1}, x_i]$ there are both rational and irrational points.

If we choose rational ξ_i on all intervals, then $I(x_i, \xi_i) = b - a$; now if all ξ_i are irrational, then $I(x_i, \xi_i) = 0$. Alternating these choices as $\Delta \to 0$, we find that the limit I does not exist. This means that the Dirichlet function is not integrable. ▲

3. Verify whether condition (1) of Theorem 2 is fulfilled for the function $f(x) = 1 + x$ on the closed interval $[-1, 4]$ and calculate $I = \int_{-1}^{4} (1+x)\,dx$ as the limit of integral sums.

△ According to Theorem 2, for an arbitrary $\varepsilon > 0$ we must indicate a partition of the interval $[-1, 4]$ for which $S - s < \varepsilon$.

We divide the interval $[-1, 4]$ into n equal parts. On each subinterval $[x_{i-1}, x_i] = \left[-1 + \frac{5(i-1)}{n}, -1 + \frac{5i}{n}\right]$ the continuous function $1 + x$ attains the greatest lower bound at the left endpoint of the interval and the least upper bound at the right endpoint. Therefore

$$s = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{5(i-1)}{n}\right) \cdot \frac{5}{n}$$

$$= \sum_{i=1}^{n} \frac{5}{n}(i-1)\frac{5}{n} = \frac{25}{n^2} \sum_{i=1}^{n} (i-1),$$

$$S = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{5i}{n}\right) \cdot \frac{5}{n}$$

$$= \sum_{i=1}^{n} \frac{5}{n} \cdot i \cdot \frac{5}{n} = \frac{25}{n^2} \sum_{i=1}^{n} i.$$

Hence

$$S - s = \frac{25}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} (i-1)\right) = \frac{25}{n^2} n = \frac{25}{n} < \varepsilon,$$

if $n > 25/\varepsilon$, i.e. for this number n of the points of division of the interval $[-1, 4]$ inequality (1) is

satisfied. This means, according to Theorem 2, that the integral $I = \int_{-1}^{4} (1+x)\,dx$ exists. To calculate it as the limit of integral sums, we can consider any sequence of integral sums for which $\Delta \to 0$ since it follows from the existence of the integral that the limit of any sequence of integral sums exists when the mesh size of the partition approaches zero and is equal to I.

We take, for instance, a sequence of integral sums which corresponds to the partition of the interval $[-1, 4]$ into n equal parts ($i = 1, 2, \ldots$) and to the choice of the right endpoints of the subintervals as the points ξ_i. In this case, for the increasing function $f(x) = 1 + x$ the integral sum is equal to the upper sum $S = \sum_{i=1}^{n} \frac{25}{n^2} i$, whence we obtain

$$I = \int_{-1}^{4} (1+x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{25}{n^2} i = \lim_{n\to\infty} \frac{25n(n+1)}{2n^2} = \frac{25}{2}.$$

Thus $\int_{-1}^{4} (1+x)\,dx = 25/2$. ▲

4. Prove that the Riemann function

$$\varphi(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/n & \text{if } x = m/n, \end{cases}$$

where m and n ($n \geqslant 1$) are coprime integers, is integrable on any closed interval $[a, b]$.

△ We again use Theorem 2. We specify an arbitrary $\varepsilon > 0$ and then the function $\varphi(x)$ satisfies the inequalities

$$\frac{\varepsilon}{2(b-a)} < \varphi(x) < 1$$

only at a certain finite number N of points.

This follows from the fact that all rational points of the closed interval $[a, b]$, i.e. points of the form m/n, can be enumerated in the following order: first points of the form $m/1$, then $m/2$, then $m/3$ and so on. The corresponding values of the function $\varphi(x)$ at these points are $1/1, 1/2, 1/3, \ldots$, i.e. decrease with the passage to each

subsequent group of points, the number of points of each kind being finite. Thus the number N of the indicated points will include those points for which

$$\frac{1}{n} > \frac{\varepsilon}{2(b-a)}, \quad \text{whence } n < \frac{2(b-a)}{\varepsilon}.$$

It is clear that the number of such points is finite (let it be N).

We cover these N points with a finite system of pairwise nonintersecting closed intervals with the total sum of lengths smaller than $\varepsilon/2$. We designate the lengths of these intervals as $\Delta x_i'$. We have thus obtained a certain partition of $[a, b]$. On the subintervals with lengths $\Delta x_i'$ the oscillations ω_i' of the function $\varphi(x)$ are not larger than unity since $\forall x \in [a, b]$ $0 \leqslant \varphi(x) \leqslant 1$. We also have a certain finite number of the other subintervals (we designate their lengths as $\Delta x_i''$). The oscillations ω_i'' of the function $\varphi(x)$ on those subintervals do not exceed $\frac{\varepsilon}{2(b-a)}$. Therefore the following estimations are valid for the partition obtained:

$$S - s = \sum \omega_i \Delta x_i = \sum \omega_i' \Delta x_i' + \sum \omega_i'' \Delta x_i''$$

$$< 1 \cdot \sum \Delta x_i' + \frac{\varepsilon}{2(b-a)} \sum \Delta x_i'' < 1 \cdot \frac{\varepsilon}{2} + \frac{\varepsilon}{2(b-a)}(b-a) = \varepsilon.$$

Thus, using the specified $\varepsilon > 0$, we have found a partition of the closed interval $[a, b]$ for which $S - s < \varepsilon$; consequently, according to Theorem 2, the Riemann function $\varphi(x)$ is integrable on any closed interval $[a, b]$. ▲

5. Calculate $\int\limits_0^{\pi} \frac{dx}{\cos^2 x\,(1+\tan^2 x)}$.

△ This integral is of the type considered in the Remark to Theorem 4 since

$$f(x) = \frac{1}{\cos^2 x\,(1+\tan^2 x)}$$

$$= \begin{cases} 1 & \text{for } 0 \leqslant x < \pi/2,\ \pi/2 < x \leqslant \pi, \\ \text{is not defined} & \text{for } x = \pi/2. \end{cases}$$

Extending the definition of this function to the point $\pi/2$, say, by continuity, i.e. setting $f(\pi/2) = 1$, we get $f(x) \equiv 1$ $\forall x \in [0, \pi]$, and, consequently, the required integral is equal to π. ▲

IV. Problems and Exercises for Independent Work

1. For given functions, on the indicated closed intervals, find the upper S and the lower s Darboux sum when the interval is divided into n equal parts: (a) $f(x) = x^3$, $2 \leqslant x \leqslant 3$, (b) $f(x) = 2^x$, $0 \leqslant x \leqslant 10$.

2. Calculate the following definite integrals as the limits of integral sums:

(a) $\int\limits_{-1}^{2} x^2\,dx$ (it is convenient to divide the interval $[-1, 2]$ into equal parts),

(b) $\int\limits_{1}^{2} \frac{dx}{x^2}$ (it is convenient to choose $\xi_i = \sqrt{x_i \cdot x_{i+1}}$),

(c) $\int\limits_{2}^{3} x^m\,dx$ (it is convenient to choose the points of division x_i such that they form a geometric progression).

3. Prove that the function $f(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$ for $x \neq 0$, $f(0) = 0$, is integrable on the closed interval $[0, 1]$.

4. Prove that the function $f(x) = \operatorname{sgn}\left(\sin \frac{\pi}{x}\right)$ is integrable on the closed interval $[0, 1]$.

8.2. Properties of the Definite Integral

I. Fundamental Concepts and Theorems

1. Properties of the definite integral

1°. By definition, $\int\limits_{a}^{a} f(x)\,dx = 0$.

2°. By definition, $\int\limits_{a}^{b} f(x)\,dx = -\int\limits_{b}^{a} f(x)\,dx$.

3°. **Linearity of the integral.** If $f(x)$ and $g(x)$ are integrable on $[a, b]$ and α and β are any real numbers, then the function $\alpha f(x) + \beta g(x)$ is also integrable on

$[a, b]$ and

$$\int_a^b \alpha f(x) + \beta g(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \beta \int_a^b g(x)\,\mathrm{d}x.$$

4⁰. If $f(x)$ is integrable on $[a, b]$, then the function $|f(x)|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x \quad (a < b).$$

5⁰. If $f(x)$ and $g(x)$ are integrable on $[a, b]$, then the function $f(x)\,g(x)$ is also integrable on $[a, b]$.

6⁰. If $f(x)$ is integrable on $[a, b]$, then it is integrable on any closed interval $[c, d] \subset [a, b]$.

7⁰. **Additivity of the integral.** If $f(x)$ is integrable on $[a, c]$ and on $[c, b]$, then it is also integrable on $[a, b]$, and

$$\int_a^b f(x)\,\mathrm{d}x + \int_a^b f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x.$$

In that case the position of the point c may be arbitrary relative to a and b.

In properties 8⁰-10⁰ we assume that $a < b$.

8⁰. If $f(x)$ is integrable on $[a, b]$ and $f(x) \geqslant 0$, then $\int_a^b f(x)\,\mathrm{d}x \geqslant 0$.

9⁰. If $f(x)$ and $g(x)$ are integrable on $[a, b]$ and $f(x) \geqslant g(x)\ \forall x \in [a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \geqslant \int_a^b g(x)\,\mathrm{d}x$.

10⁰. If $f(x)$ is continuous on $[a, b]$, $f(x) \geqslant 0$, and $f(x) \not\equiv 0$ on $[a, b]$, then $\exists K > 0$ such that $\int_a^b f(x)\,\mathrm{d}x \geqslant K$.

2. Mean value formulas

Theorem 6. *Assume that $f(x)$ and $g(x)$ are integrable on $[a, b]$, $g(x) \geqslant 0$ $(g(x) \leqslant 0)$ $\forall x \in [a, b]$, $M = \sup\limits_{[a, b]} f(x)$, $m = \inf\limits_{[a, b]} f(x)$. Then there is a number*

$\mu \in [m, M]$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx. \qquad (1)$$

Corollary 1. *If we set* $g(x) = 1$ *in formula* (1), *then*

$$\int_a^b f(x)\, dx = \mu (b-a), \quad \text{where } \mu \in [m, M]. \qquad (2)$$

The number $\mu = \frac{1}{b-a} \int_a^b f(x)\, dx$ is the *mean value of the function* $f(x)$ on the closed interval $[a, b]$.

Corollary 2. *If the conditions of Theorem 6 are fulfilled and the function* $f(x)$ *is continuous, then* $\exists \xi \in [a, b]$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx. \qquad (3)$$

Corollary 3. *If* $f(x)$ *is continuous on* $[a, b]$, *then* $\exists \xi \in [a, b]$ *such that*

$$\int_a^b f(x)\, dx = f(\xi)(b-a). \qquad (4)$$

II. Control Questions and Assignments

1. Enumerate the properties of the definite integral
2. Does the integrability of a sum imply the integrability of the summands? Give examples to substantiate the answer.
3. Consider similar questions concerning the difference, the product and the quotient of two functions.
4. Is the sum of two functions integrable if one summand is integrable and the other is not?
5. Consider similar questions concerning the difference, the product and the quotient of two functions.
6. Is the sum of two nonintegrable functions integrable? Give examples to substantiate the answer.
7. Consider similar questions concerning the difference, the product and the quotient of two nonintegrable functions.

8. It is known that $|f(x)|$ is an integrable function. What can you say of the integrability of $f(x)$? Give examples.

9. Assume that $f(x)$ is integrable on $[a, c]$ and nonintegrable on $[c, b]$. What can you say of its integrability on $[a, b]$?

10. It is known that $\int_a^b f(x)\,dx \geqslant 0$. Does it follow that $f(x) \geqslant 0$ $\forall x \in [a, b]$? Give examples.

11. It is known that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$. Does it follow that $f(x) \geqslant g(x)$ $\forall x \in [a, b]$? Give examples.

12. Give several variants of the mean value formula. Under what conditions does it hold true?

III. Worked Problems

1. Prove that the sum, the product and the quotient of two nonintegrable functions can be integrable.

△ Let $f(x) = 2 + D(x)$, $g(x) = 2 + D(x)$, where

$$D(x) = \begin{cases} 0 & \text{if } x \text{ is an irrational number,} \\ 1 & \text{if } x \text{ is a rational number} \end{cases}$$

(i.e. $D(x)$ is a Dirichlet function).

Recall that the function $D(x)$ is nonintegrable (see Example 2 in 8.1). The function $f(x) = 2 + D(x)$ is not integrable either. Indeed, if we assume that $f(x)$ is integrable, then the difference of two integrable functions $f(x) - 2 = D(x)$ must be integrable according to property 3°, but this contradicts the fact that $D(x)$ is nonintegrable. Since $g(x) = f(x)$, it follows that $g(x)$ is nonintegrable. Let us consider the function

$$h(x) = \frac{1}{g(x)} = \begin{cases} 1/2 & \text{if } x \text{ is an irrational number,} \\ 1/3 & \text{if } x \text{ is a rational number.} \end{cases}$$

This function is not integrable either. The proof is similar to the proof of the nonintegrability of a Dirichlet function.

We set up the sum, the product and the quotient of nonintegrable functions:

$$F_1(x) = f(x) + (-g(x)) \equiv 0,$$
$$F_2(x) = f(x)\, h(x) \equiv 1,$$
$$F_3(x) = f(x)/g(x) \equiv 1.$$

Being constant, the functions F_1, F_2 and F_3 are integrable on any closed interval $[a, b]$. Thus the integrability of a sum or a product does not imply the integrability of the summands or factors. ▲

2. Prove that the product of the integrable function $f(x)$ by the nonintegrable function $g(x)$ may be (a) an integrable, (b) a nonintegrable function.

△ (a) Let us consider, for example, an integrable function $f(x) \equiv 0$ and a nonintegrable Dirichlet function $D(x)$ on $[a, b]$. Since $f(x)\, D(x) \equiv 0$, it follows that $f(x)\, D(x)$ is an integrable function on $[a, b]$.

(b) Let $f(x) \equiv 2$ and $g(x) = D(x)$ on $[a, b]$. Then $f(x)\, g(x) = 2D(x)$ is a function nonintegrable on $[a, b]$. ▲

3. Find the mean value of the function on each of the indicated closed intervals: (a) $f(x) = \cos x$ on $[0, 3\pi/2]$, (b) $f(x) = \operatorname{sgn} x$ on $[-1, 2]$.

△ We seek the mean values μ using formula (2):

(a) $\mu = \frac{2}{3\pi} \int\limits_0^{3\pi/2} \cos x\, dx = -\frac{2}{3\pi}$. Note that the continuous function $\cos x$ assumes the value $\mu = -2/(3\pi)$, namely, $\cos \xi = -2/(3\pi)$, at the point $\xi = \arccos \left(-\frac{2}{3\pi}\right) \in \left[0, \frac{3\pi}{2}\right]$ of the closed interval $[0, 3\pi/2]$.

(b) $\mu = \frac{1}{3} \int\limits_{-1}^{2} \operatorname{sgn} x\, dx = \frac{1}{3}$. In this case the discontinuous function $\operatorname{sgn} x$ does not assume the value $\mu = 1/3$ on the closed interval $[-1, 3]$. ▲

IV. Problems and Exercises for Independent Work

5. Prove that the sum of an integrable and a nonintegrable function is a nonintegrable function.

6. Find out whether the following functions are integrable on the closed interval $[0, 1]$:

(a) $f_1(x) = x$, (b) $g_1(x) = 1/x$, (c) $f_1(x) + g_1(x)$, (d) $f_1(x)\, g_1(x)$, (e) $f_2(x) = \sqrt{x}$, (f) $f_2(x)\, g_1(x)$.

7. Let

$$f(x) = \begin{cases} 1 & \text{for } -2 \leqslant x \leqslant 0, \\ D(x) & \text{for } 0 < x \leqslant 2, \end{cases}$$

where $D(x)$ is a Dirichlet function. Is the function $f(x)$ integrable on the closed intervals $[-2, 2]$, $[-2, -1]$, $[-1, 1]$, $[1, 2]$?

8. Let there exist $\int\limits_a^b |f(x)|\, dx$. Does it follow that the function $f(x)$ is integrable on the closed interval $[a, b]$? Consider the example

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ -1 & \text{if } x \text{ is an irrational number.} \end{cases}$$

9. Assume that $f(x) = \sin x$, $g(x) = 0.5 \sin x$ and that (a) $0 \leqslant x \leqslant \pi$, (b) $0 \leqslant x \leqslant 3\pi/2$. In which case are the conditions of property 9 satisfied?

10. Find the mean value of the function on each of the indicated closed intervals:

(a) $f(x) = \sin x$ on $[0, \pi]$, $[0, 2\pi]$, $[\varphi_0, \varphi_0 + 2\pi]$, $[\varphi_0, \varphi_0 + \pi]$,

(b) $f(x) = \operatorname{sgn} x$ on $[-2, -1]$, $[-2, 1]$, $[-1, 3]$, $[-2, 2]$, $[1, 2]$.

Is the mean value of the function on each interval one of the values of the function on that interval? Explain why in some cases the answer is positive and in the other negative.

11. Find the mean value of the function on each of the indicated closed intervals:

(a) $f(x) = \sqrt{x}$ on $[0, 1]$, $[0, 10]$, $[0, 100]$,

(b) $f(x) = 10 + 2 \sin x + 3 \cos x$ on $[-\pi, \pi]$,

(c) $f(x) = \sin \sin (x + \varphi)$ on $[0, 2\pi]$.

12. Find the mean value of the velocity of a body falling freely from the altitude h with the initial velocity v_0.

13. The strength of an alternating current varies according to the law

$$i = i_0 \sin\left(\frac{2\pi t}{T} + \varphi\right),$$

where i_0 is the amplitude, t is time, T is the period and φ is the initial phase. Find the mean value of the square of the current strength:

(a) on the interval of time $[0, T]$,

(b) on the interval $[0, T/2]$ (the period of the function is $i^2(t)$),

(c) on the arbitrary interval $[0, t_0]$ and the limit of that mean value as $t_0 \to \infty$.

8.3. Newton-Leibniz Formula

I. Fundamental Concepts and Theorems

1. The antiderivative of a continuous and a piecewise-continuous function. Let the function $f(x)$ be integrable on the closed interval $[a, b]$. The function

$$F(x) = \int_a^x f(t)\,dt \quad (a \leqslant x \leqslant b)$$

is an *integral with a variable upper limit.*

Theorem 7. *The function $f(x)$ continuous on the closed interval $[a, b]$ has an antiderivative on this interval. One of the antiderivatives is a function*

$$F(x) = \int_a^x f(t)\,dt. \tag{1}$$

Remark. An integral with a variable upper limit is defined for any function $f(x)$ integrable on $[a, b]$. However, for the function $F(x)$ of form (1) to be an antiderivative for $f(x)$, it is essential that the function $f(x)$ be continuous.

Here is an example which shows that an integrable function may have no antiderivative. Let

$$f(x) = \operatorname{sgn} x = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -1 & \text{for } x < 0. \end{cases} \quad x \in [-1, 1].$$

This function is integrable on the closed interval $[-1, 1]$ since it is piecewise-continuous, but, as was pointed out in Chapter 5, has no antiderivative. Indeed, any

function of the form

$$F(x)=\begin{cases} -x+C_1 & \text{for } x<0 \\ x+C_2 & \text{for } x\geqslant 0, \end{cases}$$

where C_1 and C_2 are arbitrary numbers, has an antiderivative equal to sgn x for all $x \neq 0$. But even "the best" of these functions, i.e. the continuous function $F(x) = |x| + C$ (if $C_1 = C_2 = C$), does not have an antiderivative for $x = 0$. Therefore the function sgn x (and, in general, every piecewise-continuous function) does not have an antiderivative on any interval containing a point of discontinuity.

Here is an extended definition of an antiderivative which is suitable for piecewise-continuous functions as well.

Definition. *The function $F(x)$ is an **antiderivative** of the function $f(x)$ on the closed interval $[a, b]$ if: (1^0) $F(x)$ is continuous on $[a, b]$, (2^0) $F'(x) = f(x)$ at the points of continuity of $f(x)$.*

Remark. The function $f(x)$ continuous on $[a, b]$ is a special case of a piecewise-continuous function ("a piece of its continuity" coincides with the whole interval $[a, b]$). Therefore for a continuous function the extended definition of an antiderivative coincides with the old definition since $F'(x) = f(x)$ $\forall x \in [a, b]$ and the continuity of $F(x)$ follows from its differentiability.

Here is an example of a function which has an antiderivative in the "new" sense and has no antiderivative in the "old" sense. The function $f(x) = \operatorname{sgn} x$ had no antiderivative in the "old" sense on $[-1, 1]$ whereas in the "new" sense the function $F(x) = |x|$ is its antiderivative since it is continuous on $[-1, 1]$ and $F'(x) = f(x)$ for $x \neq 0$, i.e. everywhere except for the point of discontinuity $x = 0$.

The significance of the extended definition of the antiderivative is clear if we consider the following result which retains Theorem 7 with the "new" definition of the antiderivative for piecewise-continuous functions.

Theorem 8. *The function $f(x)$ piecewise-continuous on the closed interval $[a, b]$ has an antiderivative on this interval in the sense of the extended definition. The function $F(x) = \int\limits_a^x f(t)\,dt$ is one of the antiderivatives.*

2. Newton-Leibniz formula

Theorem 9. *For piecewise-continuous functions the Newton-Leibniz formula*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

holds true, where $F(x)$ is an antiderivative of the function $f(x)$ on $[a, b]$ in the sense of extended definition.

For example, $\int_1^2 \operatorname{sgn} x\,dx = |x|\Big|_{-1}^{2} = 2 - 1 = 1.$

3. The method of a change of variable

Theorem 10. *Let*

(1^0) *$f(x)$ be defined and continuous on $[a, b]$,*

(2^0) *$x = g(t)$ be defined and continuous, together with the derivative on $[\alpha, \beta]$, where $g(\alpha) = a$, $g(\beta) = b$ and $a \leqslant g(t) \leqslant b$.*

Then $\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,g'(t)\,dt.$

4. The method of integration by parts

Theorem 11. *If $f(x)$ and $g(x)$ have continuous derivatives on $[a, b]$, then*

$$\int_a^b f(x)\,g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx.$$

II. Control Questions and Assignments

1. What is an integral with a variable upper limit? For what integrand functions is it an antiderivative?
2. Give an extended definition of the antiderivative which is suitable for piecewise-continuous functions.
3. Under what conditions does the Newton-Leibniz formula hold true?
4. The function $f(x)$ is known to have an antiderivative on $[a, b]$. Is $f(x)$ integrable on $[a, b]$? Consider an example $f(x) = F'(x)$, where

$$F(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases} \quad x \in [-1, 1].$$

5. Enumerate the conditions under which (a) the formula for a change of variable and (b) the formula for integration by parts hold true.

6. By means of what substitutions can the integrals which contain (a) linear fractional irrational expressions and (b) quadratic irrational expressions be calculated?

7. What kinds of integrals can be conveniently calculated by means of trigonometric substitutions? Give examples.

8. What kinds of integrals are convenient for evaluation by integration by parts? Give examples.

III. Worked Problems

1. Find $\lim\limits_{x\to 0} \dfrac{\int\limits_0^x \cos(t^2)\,dt}{x}$.

△ This integral is an indeterminate form 0/0. The integral with a variable upper limit $\int\limits_0^x \cos(t^2)\,dt$ is an antiderivative of the continuous function $\cos x^2$, i.e. $\left(\int\limits_0^x \cos(t^2)\,dt\right)' = \cos(x^2)$. Therefore, applying L'Hospital's rule, we obtain

$$\lim_{x\to 0} \frac{\int\limits_0^x \cos(t^2)\,dt}{x} = \lim_{x\to 0} \frac{\cos(x^2)}{1} = 1.$$

Note that an antiderivative of $\cos(x^2)$ is not an elementary function, i.e. $\int\limits_0^x \cos(t^2)\,dt$ cannot be expressed in terms of elementary functions. This, however, has not prevented us from calculating the required limit. ▲

2. Find an antiderivative of the piecewise-continuous function

$$f(x) = \begin{cases} 1 & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geqslant 1, \end{cases} \quad x \in \mathbf{R},$$

△ One of the antiderivatives is an integral with a variable upper limit, and we can take any number, say, $x = -2$, as the lower limit of integration. Thus

$$F(x) = \int_{-2}^{x} f(t)\, dt$$

$$= \begin{cases} 0 & \text{for} \quad x \leqslant -1, \\ x+1 & \text{for} \quad -1 \leqslant x \leqslant 1, \\ 2 & \text{for} \quad x \geqslant 1, \end{cases} \text{(Fig. 15).} \ ▲$$

3. Calculate $I = \int_{0}^{\pi} \frac{\cos x\, dx}{\sqrt{1-\sin^2 x}}$.

△ **1st technique.** The integrand $f(x)$ is not defined at the point $x = \pi/2$. We divide the closed interval $[0, \pi]$

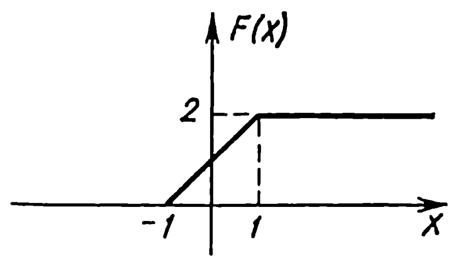


Fig. 15

in two: $[0, \pi/2]$ and $[\pi/2, \pi]$. Setting $f(\pi/2) = 1$ on the first interval, we obtain an integral of the continuous function $f \equiv 1$:

$$I_1 = \int_{0}^{\pi/2} 1\, dx = x \Big|_{0}^{\pi/2} = \pi/2.$$

On the second interval we set $f(\pi/2) = -1$ and again obtain an integral of the continuous function $f \equiv -1$:

$$I_2 = \int_{\pi/2}^{\pi} (-1)\, dx = -x \Big|_{\pi/2}^{\pi} = -\pi/2.$$

The final result is $I_1 + I_2 = 0$.

2nd technique. We use the extended definition of the antiderivative. The function $F(x)$ which satisfies this

definition has the form

$$F(x) = \begin{cases} x & \text{for } 0 \leqslant x \leqslant \pi/2, \\ \pi - x & \text{for } \pi/2 \leqslant x \leqslant \pi. \end{cases}$$

Indeed, $F(x)$ is continuous on $[0, \pi]$ and $F'(x) = f(x)$ $\forall x \in [0, \pi]$, $x \neq \pi/2$, i.e. $F'(x) = f(x)$ at the points of continuity of $f(x)$. (Recall that $x = \pi/2$ is a point of discontinuity of $f(x)$.)

In accordance with the Newton-Leibniz formula valid for piecewise-continuous functions and for the extended definition of the antiderivative, we obtain

$$I = \int_0^{\pi} f(x)\,dx = F(x)\Big|_0^{\pi} = \pi - x\,|_{x=\pi} - x\,|_{x=0} = 0. \quad \blacktriangle$$

The following two examples show that a formal application of the Newton-Leibniz formula (i.e. the use of this formula without due account of the conditions of its applicability) may lead to errors in the result.

4. Consider an integral $\int_0^1 \frac{dx}{2\sqrt{x}}$. Taking the function $F(x) = \sqrt{x}$ as an antiderivative of the integrand function $f(x) = 1/(2\sqrt{x})$ and using formally the Newton-Leibniz formula, we obtain

$$\int_0^1 \frac{dx}{2\sqrt{x}} = \sqrt{x}\Big|_0^1 = 1.$$

However, this result is incorrect since the function $f(x) = 1/(2\sqrt{x})$ is unbounded on $[0, 1]$ and, consequently, the integral $\int_0^1 \frac{dx}{2\sqrt{x}}$ does not exist.

5. Consider an integral

$$I = \int_{-1}^{1} \frac{d}{dx}\left(\arctan\frac{1}{x}\right) dx.$$

At first sight the function $\arctan(1/x)$ may seem to be an antiderivative of the integrand function $\frac{d}{dx}\left(\arctan\frac{1}{x}\right)$ and then, from the Newton-Leibniz

formula, we obtain

$$I = \arctan \frac{1}{x} \Big|_{-1}^{1} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.$$

However, this result is incorrect since the function $\arctan (1/x)$ is not an antiderivative of $\frac{d}{dx}\left(\arctan \frac{1}{x}\right)$ on the interval $[-1, 1]$. Indeed, Fig. 16*a* shows the graph of the function $\arctan (1/x)$. It can be seen that the function

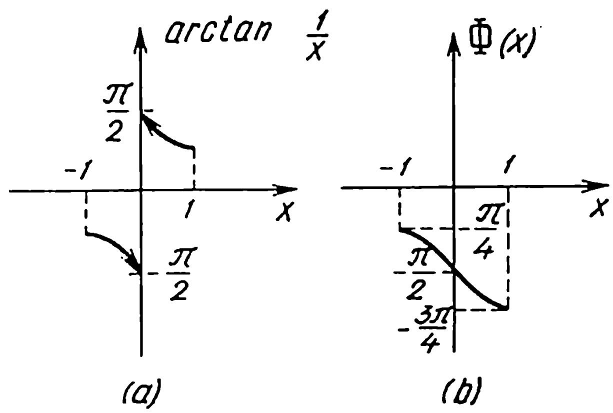


Fig. 16

has a discontinuity of the first kind at the point $x = 0$ whereas the derivative must be continuous at all points according to the definition itself.

To calculate the integral I, we note that the integrand function

$$\frac{d}{dx}\left(\arctan \frac{1}{x}\right) = \begin{cases} -\dfrac{1}{1+x^2} & \text{for } x \neq 0, \\ \text{is not defined} & \text{for } x = 0. \end{cases}$$

Extending the definition of this function to the point $x = 0$ by continuity, we get a continuous function

$$f(x) = -\frac{1}{1+x^2}, \quad x \in [-1, 1].$$

The function $F(x) = -\arctan x$ is an antiderivative of $f(x)$ and, therefore, from the Newton-Leibniz formula, we have

$$I = -\arctan x \Big|_{-1}^{1} = -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}.$$

Note that we can also construct an antiderivative for $f(x)$ using the function arctan $(1/x)$, namely,

$$\Phi(x)=\begin{cases}\arctan(1/x) & \text{for } -1\leqslant x<0,\\ -\frac{\pi}{2} & \text{for } x=0,\\ \arctan(1/x)-\pi & \text{for } 0<x\leqslant 1.\end{cases}$$

The graph of $\Phi(x)$ is shown in Fig. 16*b*. From the Newton-Leibniz formula we obtain

$$I=\Phi(x)\Big|_{-1}^{1}=-\frac{3\pi}{4}-\left(-\frac{\pi}{4}\right)=-\frac{\pi}{2}.$$

6. Using an appropriate change of variable, calculate

$$I=\int_0^a x^2\sqrt{a^2-x^2}\,dx.$$

We set $x=a\sin t$, $0\leqslant t\leqslant \pi/2$. Such a change of variable satisfies all conditions of Theorem 4. Since $\sqrt{a^2-x^2}=a\cos t$, $dx=a\cos t\,dt$, it follows that

$$I=a^4\int_0^{\pi/2}\sin^2 t\cos^2 t\,dt=\frac{a^4}{4}\int_0^{\pi/2}\sin^2 2t\,dt$$

$$=\frac{a^4}{8}\int_0^{\pi/2}(1-\cos 4t)\,dt=\frac{a^4}{8}\left(t-\frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2}=\frac{\pi a^4}{16}. \ \blacktriangle$$

The example that follows shows that a formal application of the formula for a change of variable (without due account of the conditions for its applicability) may lead to an incorrect result.

7. If we perform a formal change of variable $x=1/t$ in the integral $I=\int_{-1}^{1}\frac{dx}{1+x^2}=\frac{\pi}{2}$ and write

$$I=\int_{-1}^{1}\frac{1}{1+\frac{1}{t^2}}\left(-\frac{1}{t^2}\right)dt=-\int_{-1}^{1}\frac{dt}{1+t^2}=-\frac{\pi}{2}, \quad (2)$$

we evidently get a wrong result.

The error is due to the fact that the variation of x on the closed interval $[-1, 1]$ is associated with the variation of $t = 1/x$ not on the interval $[-1, 1]$, as is written in relation (2), but on the union of the half-lines $(-\infty, -1]$ and $[1, +\infty)$. Thus the indicated change of variable does not satisfy the requirements of Theorem 4.

8. Calculate $I = \int\limits_0^{2\pi} \frac{dx}{1+0.5\cos x}$.

$\triangle$ The integrand function $f(x) = \frac{1}{1+0.5\cos x}$ is continuous on the closed interval $[0, 2\pi]$ and, consequently, has an antiderivative. An appropriate change of variable for finding an antiderivative of the function $f(x)$ is $t = \tan(x/2)$ (see Chapter 5). However, when we seek the integral I, such a change of variable does not satisfy the conditions of Theorem 4 since the variation of t on some interval does not correspond to the variation of x on the interval $[0, 2\pi]$, i.e. $t = \tan(x/2) \to +\infty$ $(-\infty)$ as $x \to \pi - 0$ $(\pi + 0)$. Therefore we use the indicated change of variable to find the antiderivative of the integrand function. We consider the indefinite integral $\int \frac{dx}{1+0.5\cos x}$. The change of variable $t = \tan(x/2)$ is permissible for it on each of the intervals $0 \leqslant x < \pi$ and $\pi < x \leqslant 2\pi$.

In the first case the inverse function is $x = 2\arctan t$ $(0 \leqslant t < +\infty)$, in the second case it is $x = 2(\pi + \arctan t)$ $(-\infty < t \leqslant 0)$. In each case $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$, and we obtain

$$\Phi(x) = \frac{dx}{1+0.5\cos x} = 4\int \frac{dt}{3+t^2}\Big|_{t=\tan(x/2)} + C$$

$$= \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C.$$

For any constant C, the function $\Phi(x)$ is an antiderivative of $f(x) = \dfrac{1}{1+\frac{1}{2}\cos x}$ on the intervals $[0, \pi)$ and $(\pi, 2\pi]$. Since it has a discontinuity of the first kind at the point $x = \pi$, i.e. $\Phi(\pi+0) - \Phi(\pi-0) = -\frac{4\pi}{\sqrt{3}}$, it follows that $\Phi(x)$ is not an antiderivative of

$f(x)=\frac{1}{1+0.5\cos x}$ on the whole closed interval $[0, 2\pi]$. However, using $\Phi(x)$, we can now easily construct an antiderivative for $f(x)$ on the whole interval $[0, 2\pi]$. We set

$$F(x)=\begin{cases} \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) & \text{for } 0\leqslant x<\pi, \\ \frac{2\pi}{\sqrt{3}} & \text{for } x=\pi, \\ \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+\frac{4\pi}{\sqrt{3}} & \text{for } \pi<x\leqslant 2\pi. \end{cases}$$

We have thus taken $C=0$ on $[0, \pi]$, extended the definition of $\Phi(x)$ (for $C=0$) to the point $x=\pi$ by continuity from the left and have taken $C=4\pi/\sqrt{3}$ on $(\pi, 2\pi)$. We have got a function $F(x)$ whose derivative is equal to the function $f(x)$ at all points of the interval $[0, 2\pi]$, the point $x=\pi$ inclusive (prove this fact for the point $x=\pi$ yourself!), i.e. $F(x)$ is an antiderivative of $f(x)$ on $[0, 2\pi]$. From the Newton-Leibniz formula we have

$$I=F(x)\Big|_0^{2\pi}=F(2\pi)-F(0)=\frac{4\pi}{\sqrt{3}}-0=\frac{4\pi}{\sqrt{3}}. \quad \blacktriangle$$

Remark. We could have divided the integral I into two integrals $I=\int_0^{\pi} f(x)\,dx+\int_{\pi}^{2\pi} f(x)\,dx$ and use the fact that the antiderivative of $f(x)$ on $[0, \pi]$ is the function

$$F_1(x)=\begin{cases} \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) & \text{for } 0\leqslant x<\pi, \\ \frac{2\pi}{\sqrt{3}} & \text{for } x=\pi, \end{cases}$$

and on $[\pi, 2\pi]$ the function

$$F_2(x)=\begin{cases} \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) & \text{for } \pi<x\leqslant 2\pi, \\ -\frac{2\pi}{\sqrt{3}} & \text{for } x=\pi \end{cases}$$

($F_1(x)$ results from $\Phi(x)$ for $C = 0$ when the definition of $\Phi(x)$ is extended to the point $x = \pi$ by continuity from the left and $F_2(x)$ from the right). In that case, applying the Newton-Leibniz formula to each of the integrals, we obtain

$$I = F_1(x)\Big|_0^{\pi} + F_2(x)\Big|_{\pi}^{2\pi} = F_1(\pi) - F_1(0) + F_2(2\pi) - F_2(\pi)$$
$$= \frac{2\pi}{\sqrt{3}} - 0 + 0 - \left(-\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}}.$$

9. Calculate $I = \int\limits_{1/e}^{e} |\ln x|\, dx.$

$\triangle$ Dividing the integral I into the sum of integrals over the closed intervals [1/e, 1] and [1, e] (to get rid of the absolute value) and using, in each case, the formula of integration by parts, we obtain

$$I = -\int\limits_{1/e}^{1} \ln x\, dx + \int\limits_{1}^{e} \ln x\, dx$$
$$= -x \ln x\Big|_{1/e}^{1} + \int\limits_{1/e}^{1} dx + x\ln x\Big|_1^{e} - \int\limits_1^{e} dx$$
$$= -\frac{1}{e} + \left(1 - \frac{1}{e}\right) + e - (e-1) = 2\left(1 - \frac{1}{e}\right). \blacktriangle$$

10. Evaluate $I = \int\limits_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx.$

$\triangle$ We use the formula for integration by parts:

$$I = -\int\limits_0^{\pi} x \frac{d(\cos x)}{1+\cos^2 x} = -\int\limits_0^{\pi} dx\,(\arctan(\cos x))$$
$$= -x \arctan(\cos x)\Big|_0^{\pi} + \int\limits_0^{\pi} \arctan(\cos x)\, dx$$
$$= -\pi\left(-\frac{\pi}{4}\right) - 0 + I_1 = \frac{\pi^2}{4} + I_1,$$

where $I_1 = \int\limits_0^{\pi} \arctan(\cos x)\, dx.$

To find I_1, we make note of the fact that the graph of the function $f(x) = \arctan(\cos x)$ is centrally symmetric with respect to the point $(\pi/2, f(\pi/2)) = (\pi/2, 0)$. Therefore the integrals of this function over the closed intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ are equal in absolute value and opposite in sign, and, hence, their sum is zero, i.e. $I_1 = 0$. We can also establish this fact as follows: we divide the integral I_1 into two integrals over the intervals $[0, \pi/2]$ and $[\pi/2, \pi]$ respectively and make a change of variable $x = \pi - t$ in the second integral. We obtain

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \arctan(\cos x)\,\mathrm{d}x + \int_{\pi/2}^{\pi} \arctan(\cos x)\,\mathrm{d}x \\ &= \int_0^{\pi/2} \arctan(\cos x)\,\mathrm{d}x + \int_{\pi/2}^{0} \arctan(-\cos t)\,(-\mathrm{d}t) \\ &= \int_0^{\pi/2} \arctan(\cos x)\,\mathrm{d}x - \int_0^{\pi/2} \arctan(\cos t)\,\mathrm{d}t = 0. \end{aligned}$$

Thus $I_1 = 0$ and therefore $I = \pi^2/4$. ▲

IV. Problems and Exercises for Independent Work

14. Find the following derivatives:

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_a^b \sin(x^2)\,\mathrm{d}x$, (b) $\dfrac{\mathrm{d}}{\mathrm{d}b}\displaystyle\int_a^b \sin(x^2)\,\mathrm{d}x$,

(c) $\dfrac{\mathrm{d}}{\mathrm{d}a}\displaystyle\int_a^{x^2} \sin(x^2)\,\mathrm{d}x$, (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^{x^2} \sqrt{1+t^2}\,\mathrm{d}t$;

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^{x^2} \sqrt{1+x^2}\,\mathrm{d}x$, (f) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{x^2}^{x^3} \dfrac{\mathrm{d}t}{\sqrt{1+t^2}}$,

(g) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{x^2}^{x^3} \dfrac{\mathrm{d}x}{\sqrt{1+x^2}}$, (h) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{t^2}^{x^3} \dfrac{\mathrm{d}t}{\sqrt{1+t^2}}$, (i) $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{x^2}^{t^3} \dfrac{\mathrm{d}x}{\sqrt{1+x^4}}$,

(j) $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{x^2}^{x^3} \dfrac{\mathrm{d}t}{\sqrt{1+t^4}}$, (k) $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{t^2}^{x^3} \dfrac{\mathrm{d}t}{\sqrt{x^2-t^4}}$.

15. Calculate the following integrals:

(a) $\int\limits_{\sinh 1}^{\sinh 2} \frac{dx}{\sqrt{1+x^2}}$, (b) $\int\limits_0^2 |1-x|\, dx$,

(c) $\int\limits_{-1}^{1} \frac{dx}{x^2-2x\cos\alpha+1}$ $(0<\alpha<\pi)$.

16. Explain why a formal application of the Newton-Leibniz formula leads to wrong results and, using an antiderivative of a piecewise-continuous function or dividing the integration interval, calculate the following integrals:

(a) $\int\limits_0^{2\pi} \frac{dx}{\cos^2 x\,(2+\tan^2 x)}$, (b) $\int\limits_{-1}^{1} \frac{d}{dx}\left(\frac{1}{1+2^{1/x}}\right) dx$.

17. Let us calculate $\int\limits_0^2 f(x)\, dx$, where $f(x) = \begin{cases} x^2 & \text{for } 0 \leqslant x \leqslant 1, \\ 2-x & \text{for } 1 < x \leqslant 2, \end{cases}$ employing two techniques: (a) using the antiderivative of $f(x)$, constructed on the whole closed interval $[0, 2]$, (b) dividing the interval $[0, 2]$ into the intervals $[0, 1]$ and $[1, 2]$.

18. Applying the formula for integration by parts, calculate the following integrals:

(a) $\int\limits_0^{\ln 2} xe^{-x}\, dx$, (b) $\int\limits_0^{2\pi} x^2 \cos x\, dx$, (c) $\int\limits_0^1 \arccos x\, dx$.

19. Applying an appropriate change of variable, calculate the following integrals:

(a) $\int\limits_{-1}^{1} \frac{x\, dx}{\sqrt{5-4x}}$, (b) $\int\limits_0^{0.75} \frac{dx}{(x+1)\sqrt{x^2+1}}$,

(c) $\int\limits_0^{\ln 2} \sqrt{e^x-1}\, dx$, (d) $\int\limits_0^1 \frac{\arcsin \sqrt{x}}{\sqrt{x(1-x)}}\, dx$.

20. Can we calculate the integral $\int\limits_0^3 x \sqrt[3]{1-x^2}\, dx$ by the change of variable $x = \sin t$?

21. Find out whether, when calculating the integral $\int_0^1 \sqrt{1-x^2}\, dx$ by changing a variable $x = \sin t$, we can take, as the new limits of integration, the numbers (a) π and $\pi/2$, (b) 2π and $5\pi/2$, (c) π and $5\pi/2$. Calculate the integral in each case when this change of variable is permissible.

22. Prove that the following equalities are valid for the function $f(x)$ continuous on $[-l, l]$:

(a) $\int_{-l}^{l} f(x)\, dx = 2\int_0^l f(x)\, dx$ if (x) is an even function,

(b) $\int_{-l}^{l} f(x)\, dx = 0$ if $f(x)$ is an odd function.

Illustrate this fact geometrically. Find out whether these equalities hold true if $f(x)$ is integrable on $[-l, l]$ but not necessarily continuous.

23. Prove that one of the antiderivatives of an even function is an odd function and every antiderivative of an odd function is an even function.

24. Calculate the following integrals:

(a) $\int_{-1}^{1} \frac{x\, dx}{x^2+x+1}$, (b) $\int_1^e (x \ln x)^2\, dx$,

(c) $\int_0^3 \arcsin \sqrt{\frac{x}{1+x}}\, dx$, (d) $\int_0^{2\pi} \frac{dx}{(2+\cos x)(3+\cos x)}$,

(e) $\int_0^{\pi/2} \sin x \sin 2x \sin 3x\, dx$, (f) $\int_0^{\pi} (x \sin x)^2\, dx$.

25. Employing Euler's formula

$$e^{ix} = \cos x + i \sin x \quad (i \text{ is an imaginary unit number}),$$

prove that

$$\int_0^{2\pi} e^{-inx} \cdot e^{imx}\, dx = \begin{cases} 0 & \text{for } m \neq n, \\ 2 & \text{for } m = n \end{cases}$$

$\left(\text{use the equality } \int_a^b [f(x) + ig(x)]\,dx = \int_a^b f(x)\,dx + i\int_a^b g(x)\,dx\right)$.

26. Show that

$$\int_a^b e^{(\alpha+i\beta)x} dx = \frac{e^{b(\alpha+i\beta)} - e^{a(\alpha+i\beta)}}{\alpha+i\beta}$$

(use the equality $e^{(\alpha+i\beta)x} = e^{\alpha x} \cdot e^{i\beta x}$).

27. Using Euler's formulas

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}),$$

calculate the integrals

$$\text{(a)} \int_0^{\pi/2} \sin^{2m} x \cos^{2n} x\,dx, \quad \text{(b)} \int_0^{\pi} \frac{\sin nx}{\sin x}\,dx,$$

$$\text{(c)} \int_0^{\pi} \cos^n x \cos nx\,dx, \quad \int_0^{\pi} \sin^n x \sin nx\,dx.$$

8.4. Calculating the Length of Plane Curves

I. Fundamental Concepts and Formulas

1. **The length of a curve.** Consider, on the plane, a curve L defined by the parametric equations

$$x = \varphi(t), \quad y = \psi(t), \quad \alpha \leqslant t \leqslant \beta, \tag{1}$$

where $\varphi(t)$ and $\psi(t)$ are functions continuous on the closed interval $[\alpha, \beta]$, different values of $t \in [\alpha, \beta]$ being associated with different points (x, y) (i.e. there are no multiple points). Such a curve is known as a *simple (plane) open curve*.

If the points $A(\varphi(\alpha), \psi(\alpha))$ and $B(\varphi(\beta), \psi(\beta))$ coincide and the other points are not multiple, then the curve L is called a *simple closed curve*.

Let L be a simple (closed or open) curve defined by equations (1). We consider an arbitrary partition of the interval $[\alpha, \beta]$ by the points $\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta$.

It is associated with the partition of the curve L by the points $A = M_0, M_1, M_2, \ldots, M_n = B$, where $M_i = M(\varphi(t_i), \psi(t_i))$. We inscribe a polygonal line $AM_1M_2 \ldots B$ into the curve L, designate the length of the polygonal line as $l(M_i)$ and set $\Delta t = \max\limits_{1 \leqslant i \leqslant n} (t_i - t_{i-1})$.

Definition. *The number l is the* ***limit of the lengths of the polygonal lines*** *$l(M_i)$ as $\Delta t \to 0$ if $\forall \varepsilon > 0\ \exists \delta > 0$ such that the inequality $0 \leqslant l - l(M_i) < \varepsilon$ holds true for any partition of the interval $[\alpha, \beta]$ for which $\Delta t < \delta$.*

Definition. *If there is a limit of the lengths of the polygonal lines as $\Delta t \to 0$, then the curve L is said to be* ***rectifiable*** *and the number l, the* ***length of the curve*** *L (or the* ***arc length of the curve*** *L).*

2. The length of a curve represented parametrically

Theorem 12. *Let the simple curve L be defined by the parametric equations $x = \varphi(t)$, $y = \psi(t)$, $\alpha \leqslant t \leqslant \beta$, the functions $\varphi(t)$ and $\psi(t)$ having continuous derivatives on the closed interval $[\alpha, \beta]$. Then the curve L is rectifiable and its length can be found from the formula*

$$l = \int_{\alpha}^{\beta} \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt. \qquad (2)$$

The function

$$l(t) = \int_{\alpha}^{t} \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt \qquad (3)$$

expresses a *variable arc*.

3. The length of a curve in Cartesian coordinates. If a curve is defined by the equation $y = f(x)$, $a \leqslant x \leqslant b$, the function $f(x)$ having a continuous derivative on the closed interval $[a, b]$, then the length of the curve can be found from the formula

$$l = \int_{a}^{b} \sqrt{1 + f'^2(x)}\, dx. \qquad (4)$$

4. The length of a curve in polar coordinates. If a curve is defined by the equation $\rho = \rho(\varphi)$, $\varphi_1 \leqslant \varphi \leqslant \varphi_2$, with

the function $\rho(\varphi)$ having a continuous derivative on the closed interval $[\varphi_1, \varphi_2]$, then the length of the curve can be found from the formula

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2(\varphi) + \rho'^2(\varphi)}\, d\varphi.$$

If a curve is defined by the equation $\varphi = \varphi(\rho)$, $\rho_1 \leqslant \rho \leqslant \rho_2$, with the function $\varphi(\rho)$ having a continuous derivative on the closed interval $[\rho_1, \rho_2]$, then the length of the curve can be found from the formula

$$l = \int_{\rho_1}^{\rho_2} \sqrt{1 + \rho^2 \varphi'^2(\rho)}\, d\rho.$$

II. Control Questions and Assignments

1. What is a simple open (closed) curve?
2. Give the definition of the limit of the lengths of polygonal lines as $\Delta t \to 0$.
3. What is a rectifiable curve?
4. What is the length of a curve?
5. What formulas can be used to calculate the length of a curve (a) represented parametrically, (b) in Cartesian coordinates, (c) in polar coordinates?
6. Give examples of rectifiable curves.
7. Is a straight line a rectifiable curve?
8. Is a circle a simple curve?

III. Worked Problems

1. Find the length of the parabola $y = x^2$, $0 \leqslant x \leqslant 2$.

△ From formula (4) we obtain

$$l = \int_0^2 \sqrt{1 + 4x^2}\, dx = \sqrt{17} + \frac{1}{4} \ln(4 + \sqrt{17}). \quad \blacktriangle$$

2. Find the length of one segment of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$.

△ From formula (2) we obtain

$$l = a \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + \sin^2 t}\, dt = 8a. \quad \blacktriangle$$

3. Find the variable arc of the ellipse: $x = a\cos t$, $y = b\sin t$, $0 \leqslant t \leqslant 2\pi$, $a > b$.

△ From formula (3) we obtain

$$l(t) = a\int_0^t \sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2 t}\,dt = a\int_0^t \sqrt{1-\varepsilon^2\sin^2 t}\,dt,$$

where $\varepsilon = \sqrt{a^2-b^2}/a$ is the eccentricity of the ellipse. ▲

Thus the variable arc of the ellipse is expressed by an integral which is known as an elliptic integral of the second kind:

$$l(t) = a\int_0^t \sqrt{1-\varepsilon^2\sin^2 t}\,dt = aE(\varepsilon, t).$$

This integral does not possess an elementary antiderivative but is widely used in mathematics. Its name follows from the problem we have just considered.

If $t = \pi/2$, the integral equals a quarter of the length of the ellipse. In this case the elliptic integral $E(\varepsilon, \pi/2)$ is a complete elliptic integral which is written as $E(\varepsilon)$.

IV. Problems and Exercises for Independent Work

28. Find the lengths of the curves defined by the equations

(a) $y = x^{3/2}$ $(0 \leqslant x \leqslant 4)$,

(b) $x = \dfrac{y^2}{4} - \dfrac{1}{2}\ln y$ $(1 \leqslant y \leqslant e)$,

(c) $y = \ln\cos x$ $(0 \leqslant x \leqslant a < \pi/2)$,

(d) $y^2 = \dfrac{x}{2a-x}$ $(0 \leqslant x \leqslant 5a/3)$,

(e) $x^{2/3} + y^{2/3} = a^{2/3}$, (f) $x = \cos^4 t$, $y = \sin^4 t$,

(g) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leqslant t \leqslant 8\pi$; pay attention to the integration limits),

(h) $\rho = a\varphi$ $(0 \leqslant \varphi \leqslant 2\pi)$ (Archimedes spiral), (i) $\rho = a(1 + \cos\varphi)$,

(j) $\rho = a\sin^3(\varphi/3)$, (k) $\varphi = \sqrt{\rho}$ $(0 \leqslant \rho \leqslant 5)$.

29. Prove that the length of the ellipse $x = a\cos t$, $y = b\sin t$ is equal to the length of the sine curve $y = c\sin(x/b)$, $0 \leqslant x \leqslant 2\pi/b$, $c = \sqrt{a^2-b^2}$. Give a geometric illustration of this result by connecting the lengths of the ellipse and the sine curve with the section of a cylinder.

8.5. Calculating the Areas of Plane Figures

I. Fundamental Concepts and Formulas

1. The area of a plane figure. *A plane figure* is any bounded set of points of the plane.

Assume that a polygonal figure is inscribed into a given figure and a polygonal figure is circumscribed about the given figure, the polygonal figure being a figure consisting of a finite number of triangles.

The set of areas of all inscribed polygonal figures is bounded from above (by the area of any circumscribed figure) and the set of areas of all circumscribed polygonal figures is bounded from below (say, by zero).

Definition. *A plane figure is* ***squarable*** *if the least upper bound $\underline{P}$ of the set of areas of all inscribed polygonal figures is equal to the greatest lower bound $\overline{P}$ of the set of areas of all circumscribed polygonal figures.*

The number $P = \underline{P} = \overline{P}$ is the *area of the plane figure* (in the sense of Jordan).

Theorem 13 (the sufficient condition for squarability). *For a plane figure to be squarable, it is sufficient that its boundary be a rectifiable curve.*

2. The area of a plane figure in Cartesian coordinates. Assume that a plane figure is a curvilinear trapezoid

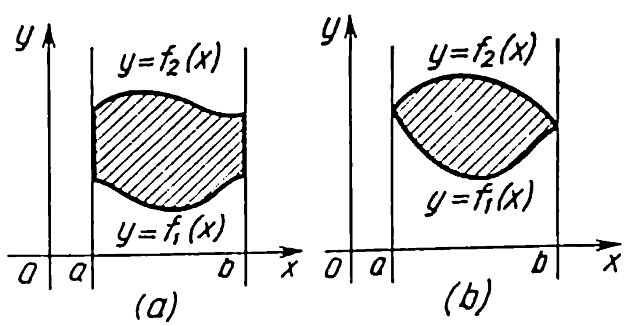


Fig. 17

bounded by the continuous curves $y = f_1(x)$, $y = f_2(x)$, $a \leqslant x \leqslant b$, where $y_1(x) \leqslant y_2(x)$, and two line segments $x = a$, $x = b$ (Fig. 17*a*). The line segments may degenerate into a point (Fig. 17*b*). Then the area of the figure

can be found from the formula

$$S=\int_a^b [f_2(x)-f_1(x)]\,\mathrm{d}x. \tag{1}$$

3. The area of a plane figure in the case of the parametric representation of its boundary. Assume that the boundary of the plane figure G is a simple closed curve defined by the parametric equations $x=\varphi(t)$, $y=\psi(t)$, $0\leqslant t\leqslant T$, and when t varies from 0 to T the point $(\varphi(t), \psi(t))$ traverses the boundary of G so that the

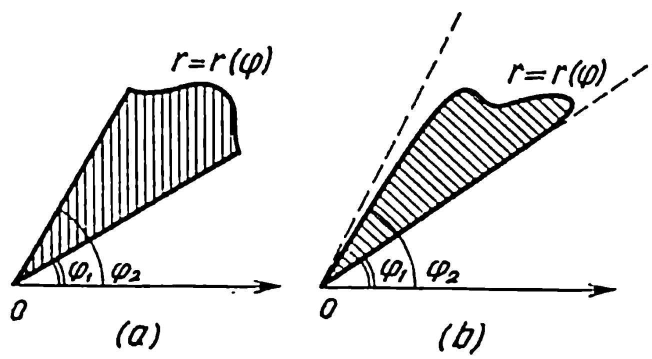


Fig. 18

figure G remains to the left of the moving point. Then the area of the figure G can be found from any of the following formulas:

$$S=-\int_0^T \psi(t)\,\varphi'(t)\,\mathrm{d}t, \tag{2}$$

$$S=\int_0^T \varphi(t)\,\psi'(t)\,\mathrm{d}t, \tag{3}$$

$$S=\frac{1}{2}\int_0^T [\varphi(t)\,\psi'(t)-\varphi'(t)\,\psi(t)]\,\mathrm{d}t. \tag{4}$$

4. The area of a plane figure in polar coordinates. Assume that a plane figure is a curvilinear sector bounded by the continuous curve $\rho=\rho(\varphi)$, $\varphi_1\geqslant\varphi\geqslant\varphi_2$, $0<\varphi_2-\varphi_1\leqslant 2\pi$, and the segments $\varphi=\varphi_1$ and $\varphi=\varphi_2$ of rays (Fig. 18*a*). The segments of rays may degenerate into a point O (Fig. 18*b*). Then the area of the figure can be found from the formula

$$S=\frac{1}{2}\int_{\varphi_1}^{\varphi_2} \rho^2(\varphi)\,\mathrm{d}\varphi. \tag{5}$$

II. Control Questions and Assignments

1. What is a plane figure?
2. What is a squarable figure?
3. What is the area of a plane figure?
4. What formulas can be used to calculate the area of a figure (a) in Cartesian coordinates, (b) in the case of the parametric representation of the boundary, (c) in polar coordinates?
5. Give examples of squarable figures.
6. Is the plane a squarable figure?
7. Is a straight line a squarable figure?

III. Worked Problems

1. Find the area of the figure bounded by the curves $y = |x - 1|$, $y = 3 - |x|$.

△ The curves meet at two points (Fig. 19). Solving the equation $3 - |x| = |x - 1|$, we find the abscissas

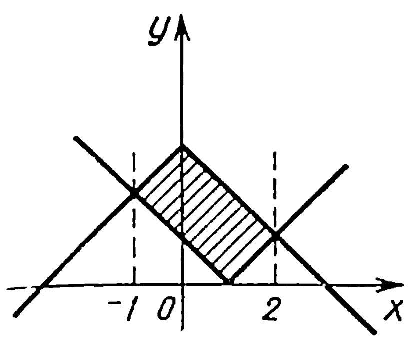


Fig. 19

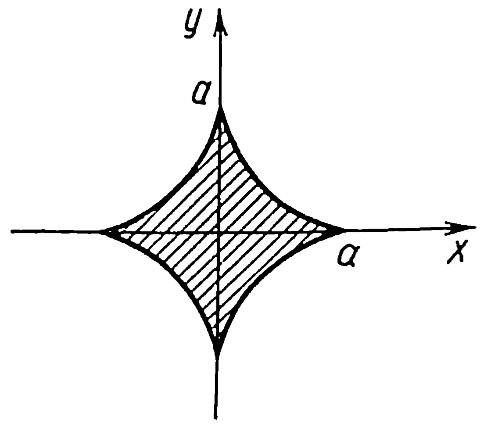


Fig. 20

of these points: $x = -1$, $x = 2$. Therefore

$$S = \int_{-1}^{2} (3 - |x| - |x - 1|)\, dx.$$

We divide the integral into three integrals over the closed intervals $[-1, 0]$, $[0, 1]$, $[1, 2]$ respectively. We obtain

$$S = \int_{-1}^{0} [(3 + x) - (1 - x)]\, dx + \int_{0}^{1} [(3 - x) - (1 - x)]\, dx$$
$$+ \int_{1}^{2} [(3 - x) - (x - 1)]\, dx = 1 + 2 + 1 = 4. \ \blacktriangle$$

2. Find the area of the figure bounded by the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ (Fig. 20).

$\triangle$ Setting $x = a\cos^3 t$, $y = a\sin^3 t$, $0 \leqslant t \leqslant 2\pi$, we arrive at the parametric equations of the astroid (the parameter t plays the part of the polar angle of the point (x, y) on the astroid). From formula (4) we get

$$S = \frac{1}{2}\int_0^{2\pi} [a\cos^3 t \cdot 3a\sin^2 t\cos t + 3a\cos^2 t\sin t \cdot a\sin^3 t]\, dt$$

$$= \frac{3}{2}a^2 \int_0^{2\pi} \sin^2 t\cos^2 t\, dt = \frac{3}{8}\pi a^2. \blacktriangle$$

Remark 1. The symmetric formula (4) has led here to an integral simpler than that which would result from the application of formula (2) or (3).

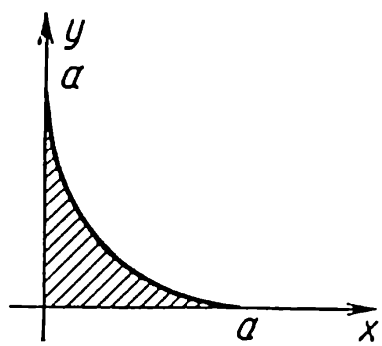


Fig. 21

Remark 2. Note that the integral taken over the closed interval $[0, \pi/2]$

$$\frac{3a^2}{2}\int_0^{\pi/2} \sin^2 t\cos^2 t\, dt = \frac{3}{32}\pi a^2$$

specifies the area of the part of the figure which lies in the first quadrant (Fig. 21), although in this case the whole boundary of the figure is no longer described by the equations $x = a\cos^3 t$, $y = a\sin^3 t$ since it contains segments of the coordinate axes. Why then have we got a correct result? The matter is that parametrically the closed interval $[a, 0]$ of the y-axis can be defined by the equations $x = 0$, $y = 2a(1 - t/\pi)$, $\pi/2 \leqslant t \leqslant \pi$ and the interval $[0, a]$ of the x-axis by the equations $x = 2a(t/\pi - 1)$, $y = 0$, $\pi \leqslant t \leqslant 3\pi/2$. Using now the com-

plete parametrization of the boundary of the figure (the parameter t varies from 0 to $3\pi/2$) and dividing the integral taken over the closed interval $[0, 3\pi/2]$ into three integrals corresponding to a curvilinear integral and to two rectilinear parts of the boundary, we find that the integrals taken over the segments of the coordinate axes vanish since on each of them one coordinate and its derivative with respect to the parameter are zero.

For the same reason formula (2) remains valid for the curvilinear trapezoid bounded by a part of the x-axis,

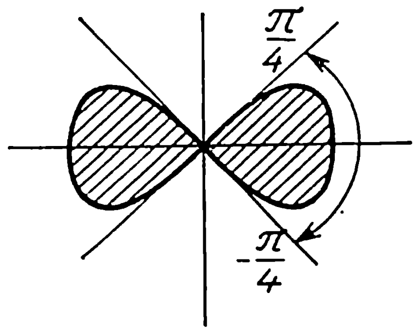


Fig. 22

two vertical line segments and a curve which has parametric equations $x = \varphi(t)$, $y = \psi(t)$, $0 \leqslant t \leqslant T$, if, when t varies from 0 to T, the point $(\varphi(t), \psi(t))$ traverses the curve so that the trapezoid remains to the left of the point. Otherwise we must put the plus sign before the integral in formula (2).

3. Find the area of the figure bounded by the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$ and the x-axis.

△ From formula (2) (where we have put the plus sign before the integral in accordance with Remark 2), we have

$$S = \int_0^{2\pi} a^2 (1 - \cos t)^2 \, dt = 3\pi a^2. \ \blacktriangle$$

4. Find the area of the figure bounded by a curve defined by the equation $\rho^2 = 2a^2 \cos 2\varphi$ in polar coordinates (lemniscate of Bernoulli).

△ Since ρ^2 is nonnegative, we find that $-\pi/4 \leqslant \varphi \leqslant \pi/4$ and $3\pi/4 \leqslant \varphi \leqslant 5\pi/4$ (Fig. 22). From formula (5)

we find the area of one of the two equal parts of the figure and double the result:

$$S=2\cdot\frac{1}{2}\cdot 2a^2\int\limits_{-\pi/4}^{\pi/4}\cos 2\varphi\, d\varphi=2a^2. \blacktriangle$$

IV. Problems and Exercises for Independent Work

30. Find the area of the figure whose boundary is defined by the following equations in Cartesian coordinates:

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad x=x_0,\ x=x_1,\ y\geqslant 0\ (-a<x_0<<x_1<a)$,

(b) $y=x^2$, $x+y=2$, (c) $y=(x+1)^2, x=\sin \pi y$, $y=0\ (0\leqslant y\leqslant 1)$,

(d) $y^2=x^2(a^2-x^2)$,

(e) $y=e^{-x}\,|\sin x\,|$, $y=0\ (x\geqslant 0)$ (assume that the area of this unbounded figure is the limit, as $A\to+\infty$, of the areas of the curvilinear trapezoids corresponding to the variation of x from 0 to A).

31. Find the area of the figure whose boundary is represented parametrically (first make a schematic drawing of the figure):

(a) $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$, $0\leqslant t\leqslant 2\pi$, $x=a$, $y\leqslant 0$ (involute of a circle),

(b) $x=a(2\cos t-\cos 2t)$, $y=a(2\sin t-\sin 2t)$.

32. Find the area of the figure whose boundary is defined by an equation in polar coordinates:

(a) $\rho=a(1+\cos\varphi)$ (cardioid), (b) $\rho=a\sin 3\varphi$ (trefoil), (c) $\rho=3+2\cos\varphi$, (d) $\rho^2+\varphi^2=1$.

33. Passing to polar coordinates, find the area of the figure whose boundary is defined by the equation

(a) $x^3+y^3=3axy$ (folium of Descartes),

(b) $(x^2+y^2)^2=2a^2xy$ (lemniscate of Bernoulli).

8.6. Calculating the Volumes of Solids

I. Fundamental Concepts and Formulas

1. The volume of a solid (in the sense of Jordan). A *solid* is any bounded set of points of space

Assume that a polyhedron is inscribed into a given solid and a polyhedron is circumscribed about the given solid, the polyhedron being a solid consisting of a finite number of triangular pyramids.

The set of volumes of all inscribed polyhedrons is bounded from above (by the volume of any circumscribed polyhedron) and the set of volumes of all circumscribed polyhedrons is bounded from below (say, by zero).

Definition. *A solid is* ***cubable*** *if the least upper bound* $\underline{V}$ *of the set of volumes of all inscribed polyhedrons is equal to the greatest lower bound* $\overline{V}$ *of the set of volumes of all circumscribed polyhedrons.*

The number $V = \underline{V} = \overline{V}$ is the *volume of the solid* (in the sense of Jordan).

2. The volume of a solid with given cross sections. Assume that each section of a cubable solid by the plane $x = \text{const}$ is a squarable figure, the area $S(x)$ of the figure being a continuous function x $(a \leqslant x \leqslant b)$. Then the volume of this solid can be found from the formula

$$V = \int_a^b S(x)\, dx. \tag{1}$$

In a particular case, when the solid results from the rotation of a curvilinear trapezoid defined by the continuous function $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, the volume of the solid can be found from the formula

$$V = \pi \int_a^b f^2(x)\, dx. \tag{2}$$

II. Control Questions and Assignments

1. What is a solid?
2. What is a cubable solid?
3. What is the volume of a solid?
4. What formula can be used to calculate (a) the volume of a solid with given cross sections, (b) the volume of a solid of revolution.
5. Give examples of cubable solids.
6. Is the plane a cubable solid?
7. Is a straight line a cubable solid?

III. Worked Problems

1. Find the volume of a solid resulting from the rotation of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ about the x-axis.

△ From formula (2) we obtain

$$V=\pi\int_{-a}^{a}\frac{b^2}{a^2}(a^2-x^2)\,dx=\frac{4}{3}\pi ab^2. \quad ▲$$

2. Find the volume of the solid bounded by the surfaces $x^2+y^2=a^2$, $z=\sqrt{3}\,y$, $z=0$ $(y\geqslant 0)$.

△ **1st technique.** Consider the sections of this solid by the planes $x=\text{const}$. The sections are right triangles with areas

$$S(x)=\frac{1}{2}y(x)z(x)=\frac{1}{2}\sqrt{a^2-x^2}\sqrt{3}\sqrt{a^2-x^2}$$
$$=\frac{\sqrt{3}}{2}(a^2-x^2).$$

From formula (1) we obtain

$$V=\frac{\sqrt{3}}{2}\int_{-a}^{a}(a^2-x^2)\,dx=\frac{2\sqrt{3}}{3}a^3.$$

2nd technique. Cutting the solid by the planes $y=\text{const}$, we get, in the sections, rectangles with areas

$$S(y)=2x(y)z(y)=2\sqrt{a^2-y^2}\sqrt{3}\,y.$$

Therefore

$$V=2\sqrt{3}\int_{0}^{a}y\sqrt{a^2-y^2}\,dy=\frac{2\sqrt{3}}{3}a^3. \quad ▲$$

IV. Problems and Exercises for Independent Work

34. Find the volume of the truncated cone whose bases are bounded by ellipses with the semi-axes A, B and a, b and whose altitude is h.

35. A solid is a set of points $M(x, y, z)$, where $0\leqslant z\leqslant 1$. In this case $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$ if z is a rational number and $-1\leqslant x\leqslant 0$, $-1\leqslant y\leqslant 0$ if z is an

irrational number. Prove that the volume of the solid does not exist although $\int_0^1 S(z)\,dz = 1$.

36. Find the volumes of the solids whose surfaces are defined by the following equations:

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=\frac{c}{a}x,\ z=0,$

(b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ z=\pm c,$

(d) $x^2+z^2=a^2,\ y^2+z^2=a^2,$

(e) $x^2+y^2+z^2=a^2,\ x^2+y^2=ax$ (Viviani's solid).

37. Find the volumes of the solids resulting from the rotation of the following curves:

(a) $y = b(x/a)^{2/3}$ $(0 \leqslant x \leqslant a)$ about the x-axis,

(b) $y = 2x - x^2$, $y = 0$ about the x-axis,

(c) $y = 2x - x^2$, $y = 0$ about the y-axis,

(d) $y = \sin x$, $y = 0$ $(0 \leqslant x \leqslant \pi)$ about the x-axis,

(e) $y = \sin x$, $y = 0$ $(0 \leqslant x \leqslant \pi)$ about the y-axis,

(f) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$ about the x-axis,

(g) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ $(0 \leqslant t \leqslant 2\pi)$ about the y-axis.

8.7. Applications of the Definite Integral in Physics

I. Fundamental Concepts and Formulas

1. Calculating the mass of a plane curve. Assume that a simple curve L is defined by the parametric equations $x = \varphi(t)$, $y = \psi(t)$, $\alpha \leqslant t \leqslant \beta$ and let $\rho(x, y)$ be the linear density of the mass at the point $(x, y) \in L$. Then the mass of the curve L can be found from the formula

$$M = \int_\alpha^\beta \rho(\varphi(t),\ \psi(t)) \sqrt{\varphi'^2(t) + \psi'^2(t)}\,dt.$$

Let the simple curve be defined by the equation in Cartesian coordinates $y = f(x)$, $a \leqslant x \leqslant b$. Then the

mass of the curve L can be found from the formula

$$M = \int_a^b \rho(x, f(x)) \sqrt{1 + f'^2(x)}\, \mathrm{d}x.$$

In particular, for $\rho \equiv 1$, the numerical value of the mass coincides with the length of the curve.

2. Calculating the moments and the coordinates of the centre of gravity of a plane curve. The static moments (or the first-order moments) of the curve L about the coordinate axes in the case of a constant linear density $\rho \equiv 1$ (geometric moments) can be found from the formulas ($x = \varphi(t)$, $y = \psi(t)$, $\alpha \leqslant t \leqslant \beta$ are the equations of the curve)

$$M_x = \int_\alpha^\beta \psi(t) \sqrt{\varphi'^2(t) + \psi'^2(t)}\, \mathrm{d}t \quad \text{(the moment about the } x\text{-axis)},$$

$$M_y = \int_\alpha^\beta \varphi(t) \sqrt{\varphi'^2(t) + \psi'^2(t)}\, \mathrm{d}t \quad \text{(the moment about the } y\text{-axis)}.$$

If the curve is given in Cartesian coordinates, $y = f(x)$, $a \leqslant x \leqslant b$, then

$$M_x = \int_a^b f(x) \sqrt{1 + f'^2(x)}\, \mathrm{d}x,$$

$$M_y = \int_a^b x \sqrt{1 + f'^2(x)}\, \mathrm{d}x.$$

The coordinates x_0 and y_0 of the centre of gravity of the curve L can be found from the formulas

$$x_0 = M_y / l, \quad y_0 = M_x / l,$$

where l is the length of the curve L.

The moments of inertia (the second-order moments) of the curve L about the coordinate axes ($\rho \equiv 1$) can be

found from the formulas

$$I_x = \int_{\alpha}^{\beta} \psi^2(t) \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt \quad \text{(about the } x\text{-axis)},$$

$$I_y = \int_{\alpha}^{\beta} \varphi^2(t) \sqrt{\varphi'^2(t) + \psi'^2(t)}\, dt \quad \text{(about the } y\text{-axis)}$$

or (in Cartesian coordinates)

$$I_x = \int_a^b f^2(x) \sqrt{1 + f'^2(x)}\, dx,$$

$$I_y = \int_a^b x^2 \sqrt{1 + f'^2(x)}\, dx.$$

3. Calculating the moments and the coordinates of the centre of gravity of a plane figure. The static moments of the figure G bounded by the continuous curves $y = f_1(x)$, $y = f_2(x)$, $a \leqslant x \leqslant b$, [where $f_1(x) \leqslant f_2(x)$] and the line segments $x = a$, $x = b$ can be calculated, in the case of the constant surface density $\rho \equiv 1$, by the formulas

$$M_x = \frac{1}{2} \int_a^b [f_2^2(x) - f_1^2(x)]\, dx \quad \text{(about the } x\text{-axis)}, \qquad (1)$$

$$M_y = \int_a^b x\, [f_2(x) - f_1(x)]\, dx \quad \text{(about the } y\text{-axis)}. \qquad (2)$$

The coordinates x_0 and y_0 of the centre of gravity of the figure can be found from the formulas

$$x_0 = M_y/S, \qquad y_0 = M_x/S, \qquad (3)$$

where S is the area of the figure G.

The moments of inertia of the figure G about the coordinate axes ($\rho \equiv 1$) can be found from the formulas

$$I_x = \frac{1}{3} \int_a^b [f_2^3(x) - f_1^3(x)]\, dx \quad \text{(about the } x\text{-axis)},$$

$$I_y = \int_a^b x^2 [f_2(x) - f_1(x)]\, dx \quad \text{(about the } y\text{-axis)}.$$

II. Worked Problems

1. Find the static moments and the coordinates of the centre of gravity of the curvilinear trapezoid bounded by the parabola $y^2 = f_2^2(x) = 2px$ and the straight lines $y = f_1(x) = 0$ and $x = 1$.

△ From formulas (1) and (2) we obtain

$$M_x = \frac{1}{2}\int_0^1 f_2^2(x)\,dx = \frac{1}{2}\cdot 2p\int_0^1 x\,dx = \frac{p}{2},$$

$$M_y = \int_0^1 x f_2(x)\,dx = \sqrt{2p}\int_0^1 x^{3/2}\,dx = \frac{2\sqrt{2p}}{5}.$$

We calculate the area of the curvilinear trapezoid:

$$S = \sqrt{2p}\int_0^1 x^{1/2}\,dx = \frac{2\sqrt{2p}}{3}.$$

And now, from formulas (3), we find the coordinates of the centre of gravity:

$$x_0 = M_y/S = 3/5, \quad y_0 = M_x/S = (3/8)\sqrt{2p}. \ ▲$$

2. Applying the second theorem of Pappus (see Exercise 44 below), find the coordinates of the centre of gravity of the plane figure G bounded by one segment of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \leqslant t \leqslant 2\pi$ and the x-axis.

△ The volume of the solid resulting from the rotation of the figure about the x-axis is

$$V = \pi\int_0^{2\pi a} y^2\,dx = \pi a^3\int_0^{2\pi}(1-\cos t)^3\,dt = 5\pi^2 a^3.$$

The area of the figure G is

$$S = \int_0^{2\pi a} y\,dx = \int_0^{2\pi} a^2(1-\cos t)^2\,dt = 3\pi a^2.$$

Let y_0 be the ordinate of the centre of gravity. According to the second theorem of Pappus, $S\cdot 2\pi y_0 = V$, whence $y_0 = 5a/6$. Since the figure G is symmetric about the straight line $x = \pi a$, it follows that the abscissa of the centre of gravity $x_0 = \pi a$. ▲

III. Problems and Exercises for Independent Work

38. Find the static moment and the moment of inertia of the semicircle of radius a relative to the diameter of the semicircle.

39. Find the static moment of the arc of the parabola $y^2 = 2px$ $(0 \leqslant x \leqslant p/2)$ relative to the straight line $x = p/2$.

40. Find the static moment and the moment of inertia of the homogeneous triangular plate with base b and altitude h relative to the base.

41. Find the moments of inertia of the homogeneous elliptic plate with semi-axes a and b relative to its principal axes.

42. Find the moment of inertia of the homogeneous circle of radius R and mass M relative to its diameter.

43. Prove the first theorem of Pappus: the area of the surface resulting from the rotation of a plane curve about an axis, which lies in its plane and does not cut it, is equal to the product of the length of the curve by the length of the circle described by the centre of gravity of the curve.

44. Prove the second theorem of Pappus: the volume of the solid resulting from the rotation of a plane figure about an axis, which lies in its plane and does not cut it, is equal to the product of the area of the figure by the length of the circle described by the centre of gravity of the figure.

45. Find the volume of the ring-shaped roll resulting from the rotation of the circle $(x-2)^2 + y^2 = 1$ about the y-axis.

46. Find the coordinates of the centre of gravity of the arc of the circle $x = a\cos\varphi, y = a\sin\varphi$ $(|\varphi| \leqslant \alpha \leqslant \pi)$.

47. Find the coordinates of the centre of gravity of the figure bounded by the parabolas $ax = y^2$, $ay = x^2$ $(a > 0)$.

48. Find the coordinates of the centre of gravity of the homogeneous half-ball of radius a.

49. Find the coordinates of the centre of gravity of the figure bounded by the curve $\rho = a(1 + \cos\varphi)$.

Chapter 9

Lebesgue Measure and Lebesgue Integral

9.1. The Measure of a Set

I. Fundamental Concepts and Theorems

1. On the sets. We say that a *one-to-one correspondence* is established between the elements of two sets if every element of the first set is associated with a certain element of the second set so that every element of the second set corresponds to only one element of the first set.

Two sets are *equivalent* if a one-to-one correspondence can be established between their elements. If two sets are equivalent, then we say that they have the same *power*.

A set is *countable* if it is equivalent to the set of natural numbers (to put it otherwise, a set is countable if its elements can be enumerated by means of natural numbers).

For example, the set of all rational numbers from the closed interval [0, 1] is countable whereas the set of all real numbers from the same interval is uncountable.

If a set is equivalent to the set of all real numbers belonging to the closed interval [0, 1], then we say that it has the *power of the continuum.*

The *union* (*sum*) of the sets $E_1, E_2, \ldots, E_n$ is the set $E = \bigcup\limits_{k=1}^{n} E_k$ all of whose elements belong to at least one of the sets E_k $(k = 1, 2, \ldots, n)$.

The union of the sets E_1 and E_2 is designated as $E_1 \cup E_2$ or $E_1 + E_2$.

The *intersection* of the sets $E_1, E_2, \ldots, E_n$ is the set $G = \bigcap\limits_{k=1}^{n} E_k$ all of whose elements belong to each of the sets E_k $(k = 1, 2, \ldots, n)$.

The intersection of the sets E_1 and E_2 is designated as $E_1 \cap E_2$ or E_1E_2.

The same definition refers to the union $\bigcup_{k=1}^{\infty} E_k$ and to the intersection $\bigcap_{k=1}^{\infty} E_k$ of a countable number of sets.

The *difference* of the sets E_1 and E_2 is the set $E = E_1 \setminus E_2$ which consists of all elements of the set E_1 which do not belong to E_2.

Let E be an arbitrary number set. The point x is an *interior* point of E if there is a neighbourhood of the point x which entirely belongs to E.

The set E is *open* if all its points are interior points. The set E is *closed* if all its limit points belong to it.

For example, the interval (a, b) is an open set whereas the interval $[a, b]$ is a closed set.

The union of a finite or countable number of open sets is an open set.

Theorem 1 (on the structure of open sets). *Any open set is the union of a finite or countable number of pairwise nonintersecting open intervals.*

2. **A number series.** Let $\{a_n\}$ be a number sequence. We formally compose an expression

$$a_1 + a_2 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

and term it a *number series* (or simply a *series*).

The numbers a_k are the *terms* of the series and the number $S_n = \sum_{k=1}^{n} a_k$ is its nth *partial sum*.

Consider a sequence $\{S_n\}$.

Definition. *If there is a limit* $\lim_{n\to\infty} S_n = S$, *then we say that the series* $\sum_{k=1}^{\infty} a_k$ *is* ***convergent*** *and the number* S *is the* ***sum of the series.***

3. **The measure of a set.** *The measure of the open interval* (α, β) (where $\beta > \alpha$) is its length $\beta - \alpha$.

Let G be a bounded open set. According to Theorem 1, it can be represented as $G = \bigcup_{k=1} (\alpha_k, \beta_k)$, where (α_k, β_k) are pairwise nonintersecting open intervals.

The *measure* μG *of the bounded open set* G is the sum of the lengths of its intervals: $\mu G = \sum_{k=1} (\beta_k - \alpha_k)$.

Note that if the number of intervals (α_k, β_k) is countable, then the sum of the lengths of the intervals is a number series $\sum_{k=1}^{\infty} (\beta_k - \alpha_k)$ with positive terms $(\beta_k - \alpha_k)$. Since the set G is bounded, the series is convergent.

Let E be an arbitrary bounded set. Let us consider various bounded open sets G which contain E. The set $\{\mu G\}$ of the measures of these sets is bounded from below (say, by the number 0) and, consequently, has inf $\{\mu G\}$.

The number $\overline{\mu} E = \inf \{\mu G\}$ is the *outer measure* of the set E.

Definition. *The set E is* ***measurable (in the sense of Lebesgue)*** *if $\forall \varepsilon > 0$ there is an open set G, containing E, for which $\overline{\mu} (G \setminus E) < \varepsilon$. In this case the outer measure of the set E is its* ***Lebesgue measure*** *and is designated as* μE, i.e. $\mu E = \overline{\mu} E$.

Remark. For the open set E this definition is equivalent to the definition given above (we can take E itself as G).

The concept of the measure of a set generalizes the concept of the length. For sufficiently simple sets (an open or closed interval) the measure coincides with the length. For more complicated sets, which have no length in its ordinary sense, the measure plays the part of the length.

Any bounded closed set is measurable.

If E is a measurable set, and $E \subset [a, b]$ then the set $\bar{E} = [a, b] \setminus E$(the *complement* of the set E with respect to the closed interval $[a, b]$) is measurable.

The union (provided that it is bounded) and the intersection of a finite or countable number of measurable sets are measurable sets. In this case, the measure of the union of a countable (or finite) number of pairwise nonintersecting sets is equal to the sum of the measures of those sets, i.e. if

$$E = \bigcup_{k=1}^{\infty} E_k, \quad E_i \cap E_j = \varnothing \quad (i \neq j),$$

then

$$\mu E = \sum_{k=1}^{\infty} \mu E_k.$$

This property is known as the *countable additivity* (or σ-additivity) of the Lebesgue measure.

II. Control Questions and Assignments

1. What is the one-to-one correspondence between the elements of two sets?

2. What sets are equivalent?

3. What set is countable?

4. Are the following sets countable: (a) the set **Q** of all rational numbers, (b) the set **R** of all real numbers?

5. What is a set of the power of the continuum? Does the set **R** of all real numbers have the power of the continuum?

6. What is the union of sets? Can the union of sets coincide with one of them? Can the union of nonempty sets be an empty set?

7. What is the intersection of sets? Can the intersection of sets coincide with one of them? Can the intersection of nonempty sets be an empty set?

8. What is the difference of two sets? Can the difference $E_1 \setminus E_2$ of nonempty sets coincide with (a) E_1, (b) E_2?

9. What is (a) an interior point of a set, (b) an open set, (c) a limit point of a set, (d) a closed set?

10. Is the set **Q** of all rational numbers (a) open, (b) closed?

11. Is the set **R** of all real numbers (a) open, (b) closed?

12. Prove that the union of a finite or countable number of open sets is an open set.

13. Prove that the intersection of a finite number of open sets is an open set and the intersection of a countable number of open sets may not be an open set.

14. Prove that the intersection of a finite or countable number of closed sets is a closed set.

15. Prove that the union of a finite number of closed sets is a closed set and the union of a countable number of closed sets may not be a closed set.

16. Formulate the theorem on the structure of open sets.

17. When do we say that a number series converges? What is the sum of a series?

18. Let all terms of the series $\sum_{k=1}^{\infty} a_k$ be nonnegative. Prove that in this case (a) the necessary and sufficient condition for convergence of the series is the boundedness of the sequence $\{S_n\}$ of its partial sums, (b) if we arbitrarily interchange the terms of the series, the sum of the series does not change.

19. What is the measure of (a) an open interval, (b) a bounded open set? Prove that the series $\mu G = \sum_{k=1}^{\infty} (\beta_k - \alpha_k)$, where (α_k, β_k) are pairwise nonintersecting open intervals from which the bounded open set G consists, is convergent.

20. What is the outer measure of a set? Does every bounded set possess an outer measure?

21. Give a definition of a measurable set and its measure.

22. Using the definition of a measurable set, prove that the closed interval $[a, b]$ is measurable and its measure $\mu [a, b] = b - a \quad (a < b)$.

23. Let the measurable set $E \subset [a, b]$. Prove that the set $\bar{E} = [a, b] \setminus E$ is measurable and $\mu \bar{E} = \mu [a, b] - \mu E$.

24. Prove that a bounded closed set is measurable.

25. What is the countable additivity of a measure? Prove that the union (if it is bounded) of the countable number of measurable sets is a measurable set.

III. Worked Problems

1. Prove that the open interval $(0, 1) = I$ and the number line $\mathbf{R}$ are equivalent sets, i.e. $\mathbf{R} \sim I$.

$\triangle$ To prove the equivalence of the sets I and $\mathbf{R}$, we must establish a one-to-one correspondence between their flements. Such a correspondence is established by the eunction

$$y = \tan\left(\pi x - \frac{\pi}{2}\right), \quad x \in I.$$

Indeed, this function puts every $x \in I$ into correspondence with some $y \in \mathbf{R}$, and since it is continuous and increases on I and, besides, $\lim\limits_{x \to 1+0} \tan\left(\pi x - \frac{\pi}{2}\right) = -\infty$, $\lim\limits_{x \to 1-0} \tan\left(\pi x - \frac{\pi}{2}\right) = +\infty$, it follows that $\forall y \in \mathbf{R}$ there is a unique $x \in I$ such that $y = \tan \times \left(\pi x - \frac{\pi}{2}\right)$. And this means that a one-to-one correspondence is established between the elements of the sets I and $\mathbf{R}$. Thus $\mathbf{R} \sim I$. $\blacktriangle$

2. Prove that the open interval $(0, 1) = I$ and the closed interval $[0, 1] = S$ are equivalent sets, i.e. $I \sim S$.

△ Let Q be the set of all rational numbers belonging to a closed interval S. This set is countable (see 2.6). We set $\overline{Q} = S \setminus Q$. Then $S = Q + \overline{Q}$. We remove the points 0 and 1 from the set Q and obtain a countable set Q_1 of rational numbers from the interval I. Evidently, $I = Q_1 + \overline{Q}$. Since Q and Q_1 are countable sets, it follows that $Q \sim Q_1$. Hence $Q + \overline{Q} \sim Q_1 + \overline{Q}$, i.e. $S \sim I$. ▲

Remark. It follows from the results of Examples 1 and 2 that the number line **R** (the set of all real numbers) has the power of the continuum.

3. Let the open set $E \subset [a, b]$. Prove that $G = [a, b] \setminus E$ is a closed set.

△ We must prove that G contains all its limit points. It follows from the definition of the difference of sets that $\forall x \in [a, b]$ either $x \in E$ or $x \in G$. Assume that x is a limit point of the set G, i.e. in any neighbourhood of the point x there are points of the set G different from x. Evidently, $x \in [a, b]$. We shall prove that $x \in G$. We assume the contrary. Then $x \in E$, and since E is an open set, there is a neighbourhood of the point x entirely belonging to E. Consequently, in this neighbourhood of the point x there are no points of the set G, but this contradicts the fact that x is a limit point of G. The contradiction obtained proves that $x \in G$ and, hence, G is a closed set. ▲

4. Prove that the set Q of all rational numbers belonging to the closed interval $[a, b]$ is measurable, and $\mu Q = 0$.

△ The set Q is countable and therefore we can enumerate its points by means of natural numbers. We specify an arbitrary $\varepsilon > 0$ and include the first point of the set G into an open interval of length $\varepsilon/2$, the second point into an interval of length $\varepsilon/2^2$, . . ., the nth point into an interval of length $\varepsilon/2^n$ and so on. The union of these intervals is an open set G whose measure

$$\mu G < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Since ε is arbitrarily small, it follows that $\bar{\mu} Q = 0$, and since $(G \setminus Q) \in G$, we find that $(G \setminus Q) \leqslant \bar{\mu} G = \mu G < \varepsilon$. By definition this means that Q is measurable, and $\mu Q =$

$\bar{\mu}Q = 0$. In this case we say that the set Q is of *measure zero*. ▲

5. Let a be an arbitrary number such that $0 < a < 1$. We construct two sets D and E by means of a countable number of steps as follows. At the first step we remove from the closed interval $[0, 1]$ an open interval E_1 of length $a/2$ which lies symmetrically with respect to the midpoint of the interval $[0, 1]$ (we shall call this open interval a middle interval). At the second step, we remove the middle intervals of length $a/8$ each from the remaining equal closed intervals. We designate the union of these intervals as E_2; the length of E_2 is $a/4$. At the third step, we remove the middle intervals of length $a/32$ each from the remaining four equal closed intervals. We designate the union of these four intervals as E_3; the length of E_3 is $a/8$. At the fourth step, we remove the middle intervals of length $a/128$ each from the remaining eight equal closed intervals and so on. Assume that

$$E = \bigcup_{k=1}^{\infty} E_k, \quad D = [0, 1] \setminus E.$$

Prove that

(a) E is an open and D is a closed set,

(b) $\mu E = a$, $\mu D = 1 - a$,

(c) the set D does not contain a closed interval in its entirety,

(d) the set D does not contain isolated points,

(e) $\forall \varepsilon > 0 \; \exists$ a set D' such that $D \subset D'$ and $0 < \mu D' - \mu D < \varepsilon$,

(f) the set D is uncountable.

△ (a) The set E is open since it is the union of open intervals, which are open sets, and the set D is closed since it is a complement of an open set with respect to a closed interval.

(b) Since the sets E_k $(k = 1, 2, \ldots)$ do not intersect, it follows, by virtue of the σ-additivity of the measure, that

$$\mu E = \sum_{k=1}^{\infty} \mu E_k = \sum_{k=1}^{\infty} \frac{a}{2^k} = a.$$

The sets D and E do not intersect and therefore

$$\mu D = \mu [0, 1] - \mu E = 1 - a.$$

(c) We assume that the set D contains a certain closed interval of length l in its entirety. Note that in the

process of construction of the set D, after the nth step of the removal of the middle intervals, the remaining part of the closed interval $[0, 1]$ consists of 2^n equal non-intersecting closed intervals. We designate the length of each of them as d_n. Evidently, $d_n \to 0$ as $n \to \infty$. For any n, the set D belongs to the union of the indicated 2^n closed intervals. Therefore the closed interval of length l, which entirely belongs to D, must entirely belong to one of the indicated closed intervals of length d_n, i.e. $l \leqslant d_n$. But this contradicts the fact that $d_n \to 0$ as $n \to \infty$. Thus the set D does not contain any closed interval in its entirety.

(d) If the set D contained an isolated point, it would follow that when constructing D, we have removed two adjoining intervals (divided by that point) from the closed interval $[0, 1]$. However, any two removed intervals are divided by some closed interval but not by a point.

(e) We set $D' = [0, 1] \setminus \bigcup\limits_{k=1}^{n} E_k$. Evidently, $D \subset D'$, and

$$D' \setminus D = \bigcup_{k=n+1}^{\infty} E_k,$$

$$\mu D' - \mu D = \sum_{k=n+1}^{\infty} \mu E_k = \sum_{k=n+1}^{\infty} \frac{a}{2^k} = \frac{a}{2^n}.$$

It follows that $\forall \varepsilon > 0 \quad \exists n$ such that there holds an inequality

$$0 < \mu D' - \mu D = \frac{a}{2^n} < \varepsilon.$$

(f) A countable set is of measure zero (see Example 4). Since $\mu D = 1 - a \neq 0$, it follows that D is an uncountable set. ▲

IV. Problems and Exercises for Independent Work

1. Test the equivalence of the following sets:
   (a) $\{1, 2, 3, 4, \ldots\}$ and $\{2, 4, 6, 8, \ldots\}$,
   (b) the closed intervals $[0, 1]$ and $[a, b]$,
   (c) the open interval (a, b) and the number line $\mathbf{R}$.
2. Prove that
   (a) an infinite subset of a countable set is also countable,

(b) the union of a finite or a countable number of countable sets is a countable set,

(c) the set of all polynomials with rational coefficients is countable,

(d) the set of points of discontinuity of a monotonic function is finite or countable.

3. Prove that the set of all real numbers belonging to the closed interval $[0, 1]$ is uncountable.

4. Prove that

(a) if A is an infinite set and B is a finite or a countable set, then $A + B \sim A$,

(b) if A is an infinite set, B is a finite or countable set and $A \setminus B$ is an infinite set, then $A \setminus B \sim A$,

(c) the set of all irrational numbers belonging to the closed interval $[0, 1]$ has the power of the continuum,

(d) every infinite set contains a part equivalent to the whole set.

5. Prove that for any sets A, B and C

(a) $(A + B)\,C = AC + BC$, (b) $A + A = A$,

(c) $AA = A$, (d) $A + BC = (A + B)(A + C)$,

(e) $A = (A \setminus B) + AB$, in particular, $A = (A \setminus B) + B$ if $B \subset A$.

6. Assume that $\forall k$: $A_k \subset E$, $\overline{A_k} = E \setminus A_k$ is a complement of A_k with respect to E (the number of sets A_k is finite or countable). Prove that $\overline{\bigcup\limits_k A_k} = \bigcap\limits_k \overline{A_k}$.

7. Let the closed set $E \subset (a, b)$. Prove that $G = (a, b) \setminus E$ is an open set.

8. Prove that the set $\overline{Q}$ of all irrational numbers belonging to the closed interval $[a, b]$ is measurable and find its measure.

9. Let E be a bounded set such that $\overline{\mu} E = 0$. Prove that E is measurable and that $\mu E = 0$.

10. Prove that every subset of the set of measure zero is of the measure zero.

9.2. Measurable Functions

I. Fundamental Concepts and Theorems

1. Definition of a measurable function. Let the function $f(x)$ be defined on a measurable set E. We shall designate as $\{x \in E: f(x) \leqslant c\}$ the set of all values of the argu-

ment x which belong to the set E and are such that $f(x) \leqslant c$ for them (c is a number).

Definition. *The function $f(x)$ is* ***measurable on the set*** *E if for any number c the set $\{x \in E: f(x) \leqslant c\}$ is measurable.*

Theorem 2. *For the function $f(x)$ to be measurable on the set E, it is necessary and sufficient that for any number c any one of the following sets be measurable*: $\{x \in E: f(x) > c\}$, $\{x \in E: f(x) \geqslant c\}$, $\{x \in E: f(x) < c\}$.

2. Some properties of measurable functions.

1^0. If the function $f(x)$ is measurable on the set E, then it is measurable on any measurable subset of the set E.

2^0. If the function $f(x)$ is measurable on the sets E_1, E_2, . . ., E_n, . . ., then it is measurable on their union $\bigcup\limits_{k=1}^{\infty} E_k$ and intersection $\bigcap\limits_{k=1}^{\infty} E_k$.

3^0. If the function $f(x)$ is defined on the set E of measure zero, then it is measurable on that set.

4^0. If the functions $f(x)$ and $g(x)$ are measurable on the set E, then the functions $f(x) + g(x)$, $f(x) - g(x)$, $f(x)\, g(x)$ and $f(x)/g(x)$ (provided that $g(x) \neq 0$) are also measurable on the set E.

5^0. A function continuous on a closed interval is measurable on that interval.

We say that a property is valid *almost everywhere* on the set E if the set of points of E on which it is not valid is of measure zero.

The functions $f(x)$ and $g(x)$, defined on a measurable set E, are *equivalent* on that set if they are equal almost everywhere on it.

The designation of the equivalence is $f(x) \sim g(x)$ on E.

For instance, the Dirichlet function

$$D(x) = \begin{cases} 0 \text{ if } x \text{ is an irrational number,} \\ 1 \text{ if } x \text{ is a rational number, } x \in [a, b], \end{cases} \tag{1}$$

is equivalent to a continuous function $g(x) \equiv 0$ on $[a, b]$ since the set of points x which belong to $[a, b]$ and on which $D(x) \neq g(x)$ is the set Q of all rational numbers from the closed interval $[a, b]$ whose measure is $\mu Q = 0$. Note that the function $D(x)$ is discontinuous at all points of $[a, b]$.

Here are another two properties of measurable functions.

6^0. If $g(x)$ is measurable on E, $f(x) \sim g(x)$ on E, then $f(x)$ is measurable on E.

7^0. **Theorem 3 (Lusin's theorem or the C-property of measurable functions).** *For the function $f(x)$ to be measurable on $[a, b]$, it is necessary and sufficient that $\forall \varepsilon > 0$ there should be a function $g(x)$, continuous on $[a, b]$, such that $\mu\{x \in [a, b]: f(x) \neq g(x)\} \leqslant \varepsilon$.*

Lusin's theorem means that any function measurable on $[a, b]$ can be made measurable by varying it on a set of an arbitrarily small measure, i.e. functions measurable in this sense are close to continuous functions.

II. Control Questions and Assignments

1. Give a definition of a measurable function.
2. Using the definition, prove that the following functions are measurable:

(a) $f(x) = c = \text{const}, \ x \in [a, b]$,

(b) $f(x) = x, \ x \in [a, b]$,

(c) $f(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant 1/2, \\ 1 & \text{for } 1/2 < x \leqslant 1. \end{cases}$

3. Formulate the theorem which expresses a necessary and sufficient condition for a function to be measurable on a set E.
4. Formulate properties 1^0-5^0 of measurable functions. Prove properties 1^0-3^0.
5. When do we say that some property is valid almost everywhere on a set E?
6. What functions are equivalent on a set E? Prove that a Dirichlet function is equivalent to a continuous function on $[a, b]$. What is the measure of the set of points of discontinuity of the Dirichlet function on $[a, b]$?
7. Formulate and prove property 6^0 of measurable functions.
8. Using properties 5^0 and 6^0, prove that if the function $f(x)$ is equivalent to a continuous function on $[a, b]$, then $f(x)$ is measurable on $[a, b]$.
9. Formulate Lusin's theorem. Use the function $f(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant 1/2, \\ 1 & \text{for } 1/2 < x \leqslant 1 \end{cases}$ as an example.

III. Worked Problems

1. Prove that the Dirichlet function $D(x)$ [see formula (1)] is measurable on $[a, b]$.

$\triangle$ Using the definition of a measurable function, we must prove that $\forall c$ the set $\{x \in [a, b]: f(x) \leqslant c\} \equiv E_c$ is measurable. We shall consider three cases.

If $c \geqslant 1$, then $E_c = [a, b]$ is a measurable set.

If $0 \leqslant c < 1$, then $E_c = \overline{Q}$, where $\overline{Q}$ is the set of all irrational numbers from $[a, b]$, is a measurable set.

If $c < 0$, then $E_c = \varnothing$, where $\varnothing$ is an empty set. An empty set is considered to be measurable: $\mu\varnothing = 0$.

Thus $D(x)$ is a measurable function. $\blacktriangle$

2. Prove property 5⁰: the function $f(x)$ continuous on $[a, b]$ is measurable on $[a, b]$.

$\triangle$ We shall prove that $\forall c$ the set $\{x \in [a, b]: f(x) \leqslant c\} \equiv E_c$ is measurable. If $E_c = \varnothing$, then E_c is measurable: $\mu E_c = \mu\varnothing = 0$. Let E_c be nonempty. We shall prove that E_c is a closed set. It will follow that E_c is measurable. Let x_0 be the limit point of the set E_c. We must prove that $x_0 \in E_c$. It is clear that $x_0 \in [a, b]$. According to the definition of a limit point, $\exists\{x_n\} \to x_0$ and $\forall n$ $x_n \in E_c$. It follows from the continuity of $f(x)$ at the point x_0 that $\{f(x_n)\} \to f(x_0)$, and since $x_n \in E_c$, it follows that $f(x_n) \leqslant c$. Hence $f(x_0) \leqslant c$, i.e. $x_0 \in E_c$, and this is what we wished to prove. $\blacktriangle$

IV. Problems and Exercises for Independent Work

11. Prove the theorem which expresses a necessary and sufficient condition for a function to be measurable on the set E.

12. Let $f(x) = \begin{cases} 0 & \text{for } x \in E, \\ 1 & \text{for } x \in D = [0, 1] \setminus E, \end{cases}$ where E and D are the sets from Example 5 in 9.1. Prove that

(a) $f(x)$ is a measurable function on $[0, 1]$,

(b) D is the set of all points of discontinuity of $f(x)$,

(c) $\forall \varepsilon > 0$ there is a function $g(x)$, continuous on $[0, 1]$, such that $\mu\{x \in [0, 1]: f(x) \neq g(x)\} < \varepsilon$.

13. Prove that the function $\varphi(x) = x$, $x \in D$, where D is the set from Example 12, is measurable on D.

14. Using Lusin's theorem, prove that the sum, the difference and the product of functions measurable on $[a, b]$ are functions measurable on $[a, b]$.

9.3. Lebesgue Integral

I. Fundamental Concepts and Theorems

1. Lebesgue integral of a bounded function. Let E be an arbitrary measurable set. The *partition* of the set E is any collection $T = \{E_k\}$ of a finite number of measurable sets $E_1, E_2, \ldots, E_n$ such that $\bigcup\limits_{k=1}^{n} E_k = E$, $E_i \cap E_j = \varnothing$ $(i \neq j)$.

Assume that a bounded function $f(x)$ is defined on the set E. For the arbitrary partition $T = \{E_k\}$ we set $M_k = \sup\limits_{E_k} f(x)$, $m_k = \inf\limits_{E_k} f(x)$ and set up two sums:

$$S_T = \sum_{k=1}^{n} M_k \mu E_k, \qquad s_T = \sum_{k=1}^{n} m_k \mu E_k.$$

The numbers S_T and s_T are the *upper* and the *lower sum* of the partition T respectively. It is evident that $\forall T$ $s_T \leqslant S_T$.

Let us consider the number sets $\{s_T\}$ and $\{S_T\}$ of various lower and upper sums. They are bounded from below by the number $m\mu E$ and from above by the number $M\mu E$, where $m = \inf\limits_{E} f(x)$, $M = \sup\limits_{E} f(x)$ and, consequently, have the greatest lower bound and the least upper bound. The numbers $\underline{I} = \sup \{s_T\}$ and $\overline{I} = \inf \{S_T\}$ are the *lower* and *upper Lebesgue integrals* respectively.

Definition. *A function $f(x)$ bounded on the measurable set E is said to be* ***Lebesgue integrable*** *on this set if $\underline{I} = \overline{I}$.*

In this case the number $\underline{I} = \overline{I}$ is a *Lebesgue integral* of the function $f(x)$ on the set E and is designated as $\int\limits_E f(x)\, d\mu(x)$.

2. The connection between the Riemann and Lebesgue integrals

Theorem 4. *Every function which is Riemann integrable on $[a, b]$ is Lebesgue integrable on $[a, b]$, the Riemann and Lebesgue integrals of such a function being equal.*

Remark. The converse statement is not true (see the Example in item III).

3. Lebesgue integrable bounded functions

Theorem 5. *For a function $f(x)$ bounded on the measurable set E to be Lebesgue integrable on this set, it is necessary and sufficient that $f(x)$ be measurable on E.*

4. Lebesgue integral as the limit of Lebesgue integral sums. Assume that $f(x)$ is a bounded measurable function on the set E, $m = \inf\limits_E f(x)$, $M = \sup\limits_E f(x)$, and let $y_1, y_2, \ldots, y_{n-1}$ be arbitrary numbers such that $m = y_0 < y_1 < y_2 < \ldots < y_n = M$.

The *Lebesgue partition* of the set E is the partition $T = \{E_k\}$, where $E_1 = \{x \in E\colon\ y_0 \leqslant f(x) \leqslant y_1\}$, $E_k = \{x \in E\colon\ y_{k-1} < f(x) \leqslant y_k\}$, $k = 2, 3, \ldots, n$.

Let ξ_k be an arbitrary point from E_k ($k = 1, 2, \ldots, n$). The number $I(E_k, \xi_k) = \sum\limits_{k=1}^{n} f(\xi_k)\, \mu E_k$ is a *Lebesgue integral sum* (if some $E_k = \varnothing$, then E_k does not contain any points ξ_k and the corresponding term in the sum is considered to be zero).

We set $\delta = \max\limits_{1 \leqslant k \leqslant n} \delta_k$, where $\delta_k = y_k - y_{k-1}$.

Theorem 6. *As $\delta \to 0$, the limit of the Lebesgue integral sums is equal to the Lebesgue integral i.e.* $\lim\limits_{\delta \to 0} I \times (E_k, \xi_k) = \int\limits_E f(x)\, d\mu(x)$.

It follows from this theorem that the Lebesgue integral can be defined as the limit of the Lebesgue integral sums as $\delta \to 0$. Such a definition of the Lebesgue integral is similar to the definition of the Riemann integral with the only difference that when composing integral sums we divide into subintervals not the domain of definition but the range of the function.

5. Properties of the Lebesgue integral

1°. $\int\limits_E d\mu(x) = \mu E$.

2°. **Linearity of the integral.** If $f(x)$ and $g(x)$ are integrable on E and α and β are arbitrary numbers, then the function $\alpha f(x) + \beta g(x)$ is integrable on E, and

$$\int\limits_E [\alpha f(x) + \beta g(x)]\, d\mu(x) = \alpha \int\limits_E f(x)\, d\mu(x) + \beta \int\limits_E g(x)\, d\mu(x).$$

3°. **Additivity of the integral.** If $f(x)$ is integrable on

E_1 and E_2, and $E_1 \cap E_2 = \varnothing$, then $f(x)$ is integrable on $E_1 \cup E_2$ and the equality

$$\int_{E_1 \cup E_2} f(x)\, d\mu(x) = \int_{E_1} f(x)\, d\mu(x) + \int_{E_2} f(x)\, d\mu(x),$$

holds true.

4^0. If $f(x)$ and $g(x)$ are integrable on E, and $f(x) \geqslant g(x)$ $\forall x \in E$, then

$$\int_E f(x)\, d\mu(x) \geqslant \int_E g(x)\, d\mu(x).$$

II. Control Questions and Assignments

1. What is the partition of the measurable set E?
2. What are the upper and lower sums of a given partition?
3. Prove that if a partition T_2 is obtained from the partition $T_1 = \{E_k\}$ by means of partitioning some E_k (i.e. by the refining the partition T_1), then $s_{T_1} \leqslant s_{T_2}$, $S_{T_1} \geqslant S_{T_2}$.
4. Prove that $s_{T_1} \leqslant S_{T_2}$, $s_{T_2} \leqslant S_{T_1}$ for any partitions T_1 and T_2.
5. What are the upper and the lower Lebesgue integral? Prove that $\underline{I} \leqslant \overline{I}$.
6. Give a definition of the Lebesgue integrable function and the Lebesgue integral.
7. What is the relationship between the Lebesgue and Riemann integrals?
8. Characterize the class of the Lebesgue integrable bounded functions.
9. What is the Lebesgue partition?
10. What is the Lebesgue integral sum?
11. Can we define the Lebesgue integral as the limit of Lebesgue integral sums? What is the common feature of and the difference between such a definition of the Lebesgue integral and the definition of the Riemann integral?
12. Enumerate the properties of the Lebesgue integral.

III. A Worked Problem

Prove that the Dirichlet function $D(x)$ (see formula (1) in 9.2) is Lebesgue integrable on $[a, b]$ and find

$$\int_{[a, b]} D(x)\, d\mu(x).$$

△ **1st technique.** Since $D(x) \geqslant 0$, we have $s_T \geqslant 0$, $S_T \geqslant 0$ for any partition T. We consider a partition T^* of the closed interval $[a, b]$ into a set Q of rational numbers and a set $\bar{Q}$ of irrational numbers. For this partition

$$S_{T^*} = \sup_Q D(x)\,\mu Q + \sup_{\bar{Q}} D(x)\,\mu\bar{Q} = 1\cdot 0 + 0\cdot(b-a) = 0.$$

Thus the set $\{S_T\}$ contains a number 0. Therefore $\bar{I} = \inf\{S_T\} = 0$.

Since all $s_T \geqslant 0$, it follows that $\underline{I} = \sup\{s_T\} \geqslant 0$, and since $\underline{I} \leqslant \bar{I}$, we get $\underline{I} = 0$.

Thus $\underline{I} = \bar{I} = 0$. It follows that the function $D(x)$ is Lebesgue integrable on $[a, b]$ and $\int\limits_{[a,b]} D(x)\,d\mu(x) = 0$.

2nd technique. For any Lebesgue partition $T = \{E_k\}$ we have

$$\mu E_1 = \mu\{x \in [a, b]:\ 0 \leqslant D(x) \leqslant y_1 < 1\}$$
$$= \mu\bar{Q} = b - a,\quad f(\xi_1) = 0\ \ \forall \xi_1 \in E_1,$$
$$\mu E_2 = \mu E_3 = \ldots = \mu E_{n-1} = \mu\varnothing = 0,$$
$$\mu E_n = \mu\{x \in [a, b]:$$
$$0 < y_{n-1} < D(x) \leqslant 1\} = \mu Q = 0,\quad f(\xi_n) = 1$$
$$\forall \xi_n \in E_n.$$

Therefore $I(E_k, \xi_k) = f(\xi_1)\,\mu\bar{Q} + f(\xi_n)\,\mu Q = 0$, i.e. any Lebesgue integral sum is zero. Consequently, $\lim\limits_{\delta\to 0} I(E_k, \xi_k) = 0$, i.e. the function $D(x)$ is integrable on $[a, b]$ and $\int\limits_{[a,b]} D(x)\,d\mu(x) = 0$. ▲

Remark. It is known that the function $D(x)$ is Riemann nonintegrable on $[a, b]$. Thus a Lebesgue integrable function may be nonintegrable in the sense of Riemann.

IV. Problems and Exercises for Independent Work

15. Let $f(x)$ be the function from Exercise 12. Prove that $f(x)$ is Lebesgue integrable but not Riemann integrable on $[0, 1]$.

16. For the function $f(x)$ from Exercise 15 set up Lebesgue integral sums and calculate $\int\limits_{[0,1]} f(x)\,d\mu(x)$.

17. Prove that the function $\varphi(x)$ from Exercise 13 is Lebesgue integrable on the set D and calculate $\int\limits_D \varphi(x)\, d\mu(x)$.

18. Prove that if the function $f(x) = 0$ almost everywhere on the measurable set E, then it is integrable, and $\int\limits_E f(x)\, d\mu(x) = 0$.

19. Prove that if the bounded functions $f(x)$ and $g(x)$ are equivalent on the set E and the function $f(x)$ is integrable on E, then the function $g(x)$ is also integrable on E, and $\int\limits_E g(x)\, d\mu(x) = \int\limits_E f(x)\, d\mu(x)$.

20. Prove that every function which is Riemann integrable on $[a, b]$ is Lebesgue integrable on $[a, b]$ and the Riemann and Lebesgue integrals of such a function are equal.

21. Prove the sufficiency in Theorem 5, i.e. prove that a function which is bounded and measurable on the set E is Lebesgue integrable of that set.

22. Prove that the following test of the Lebesgue integrability of functions (similar to the test of the Riemann integrability) holds true: for a function bounded on the measurable set E to be Lebesgue integrable on that set, it is necessary and sufficient that $\forall \varepsilon > 0$ there should be a Lebesgue partition T of the set E for which the inequality $S_T - s_T < \varepsilon$ holds ture.

23. Prove that if the function $f(x)$ is bounded and measurable on the set E, then the limit of its Lebesgue integral sums, as $\delta \to 0$ $[\delta = \max\limits_{1 \leqslant k \leqslant n} (y_k - y_{k-1})]$ is equal to the Lebesgue integral of the function $f(x)$ over the set E.

Chapter 1

1. ● Use the method of indirect proof. 4. Assume the contrary, i.e. let the fraction of the form $a_0.a_1a_2 \ldots a_k\overline{9}$ be the result of the division of the natural number m by the natural number n. Then $\forall p \in \mathrm{N}$, $p > k$, the number m/n must satisfy the inequalities

$$a_0.\underbrace{a_1 \cdots a_k 99 \cdots 9}_{p \text{ digits}} < \frac{m}{n} < a_0.a_1 \cdots a_k + \frac{1}{10^k}.$$

Consequently,

$$0 < a_0.a_1 \cdots a_k + \frac{1}{10^k} - \frac{m}{n}$$
$$< a_0.a_1 \cdots a_k + \frac{1}{10^k} - a_0.\underbrace{a_1 \cdots a_k 99 \cdots 9}_{p \text{ digits}} = \frac{1}{10^p},$$

and this is impossible (explain why). The contradiction obtained proves the initial statement. 5. ● Use the results of Examples 1 and 2. 15. Assume that x is any positive number and x_1 and y_1 are any rational numbers which satisfy the inequalities

$$0 < x_1 \leqslant x, \qquad 0 < y_1 \leqslant 1. \tag{*}$$

Then, according to the definition of the product of positive numbers, we have $x \cdot 1 = \sup M$, where $M = \{(x_1y_1)_r\}$. It follows from (*) that $\forall x_1, y_1\ 0 < (x_1y_1)_r \leqslant x_1 \leqslant x$, i.e. x is the upper bound of the set M. Let $\tilde{x} < x$. It is shown in Example 1 in 1.1 that there is a rational number x^* such that $\tilde{x} < x^* < x$. We take $x_1 = x^*$, $y_1 = 1$. Then $(x_1y_1)_r = x^* \cdot 1 = x^*$ and, consequently, $(x_1y_1)_r > \tilde{x}$. Thus both conditions of the definition of the least upper bound of a set are fulfilled for the number x, i.e. $\sup M = x$. Thus $\forall x > 0$: $x \cdot 1 = \sup M = x$, whence $x \cdot 1 = x$. If $x = 0$, then, according to the rule of multiplication of rational numbers, $0 \cdot 1 = 0$. If $x < 0$, then, in accordance with the definition of the product of real numbers, $x \cdot 1 = -|x| \cdot 1$. But, as we have just proved, $|x| \cdot 1 = |x|$, i.e. $x \cdot 1 = -|x| = x$. 33. ● For $x_1x_2 \ldots x_n = 0$ the statement is obvious. For $x_1x_2 \ldots x_n \neq 0$ set $y_k = x_k/\sqrt[n]{x_1x_2 \ldots x_n}$ $(k = 1, 2, \ldots, n)$ and use the result of Example 2 in 1.4.

Chapter 2

1. (a) Yes, it is, (b) no, it isn't, (c) no, it isn't, (d) yes, it is, (e) no, it isn't. 4. No, it does not. 5. ● It is sufficient to prove that the sequence $\{x_n\}$ is unbounded. 14. ● If $\lim\limits_{n\to\infty} x_n = +\infty$, then $\exists A > 0$ and a number N which are such that $x_n > A$ $\forall n > N$. The sequence $x_1, x_2, \ldots, x_N$ includes the least number. 15. (a) $x_4 = x_5 = -120$, (b) $x_{10} = 20$. 20. (a) It converges if $\alpha > 0$, $\beta > 0$, $\alpha \leqslant \beta$ or $\alpha \leqslant 0$, β is arbitrary, (b) it converges if $\gamma \leqslant 3/2$. 21. (a) 0, (b) 0, (c) 1/3. 23. (a) 1/2, (b) 1/3, (c) 1. ● Represent the fraction $\dfrac{1}{k(k+1)}$ in the form $\dfrac{1}{k} - \dfrac{1}{k+1}$ $(k = 1, 2, \ldots, n)$, (d) 1/4. ● Represent $S_n = \dfrac{1}{1\cdot 2\cdot 3} + \ldots$ $\ldots + \dfrac{1}{n(n+1)(n+2)}$ in the form $S_n = A + \dfrac{B}{n+1} + \dfrac{C}{n+2}$ $\forall n$. 27. Since $\forall \varepsilon > 0$ $\lim\limits_{n\to\infty} \dfrac{(1/\varepsilon)^n}{n!} = 0$, it follows that $\forall \varepsilon > 0$ $\exists N$ such that $\forall n > N$ there holds an inequality $(1/\varepsilon)^n/n! < 1$, or $1/\sqrt[n]{n!} < \varepsilon$. 29. $(1+\sqrt{1+4a})/2$. 30. (a) $(a+2b)/3$. (b) $\sqrt{a}$, (c) $-\sqrt{a}$. 33. ● (d) Prove that (1°) the set U of limit points of the sequence is bounded, (2°) if $\inf U = c$, $\sup U = b$, then $a, b \in U$, (3°) $c = \varliminf\limits_{n\to\infty} x_n$, $b = \varlimsup\limits_{n\to\infty} x_n$. 35. (a) The limit points are 2, −2; $\varlimsup\limits_{n\to\infty} x_n = 2$, $\varliminf\limits_{n\to\infty} = -2$, (b) the limit points are 0, 1, 2; $\varlimsup\limits_{n\to\infty} x_n = 2$, $\varliminf\limits_{n\to\infty} x_n = 0$, (c) the limit points are −4, 0, 2, 6; $\varlimsup\limits_{n\to\infty} x_n = 6$, $\varliminf\limits_{n\to\infty} x_n = -4$, (d) the limit points are −1/2, 1; $\varlimsup\limits_{n\to\infty} x_n = 1$, $\varliminf\limits_{n\to\infty} x_n = -1/2$, (e) the limit point is 1; $\varlimsup\limits_{n\to\infty} x_n = +\infty$, $\varliminf\limits_{n\to\infty} x_n = -\infty$, (f) the limit points are 0, 1; $\varlimsup\limits_{n\to\infty} x_n = 1$, $\varliminf\limits_{n\to\infty} x_n = 0$; (g) the limit points are $-e - \dfrac{1}{\sqrt{2}}$, $-e + \dfrac{1}{\sqrt{2}}$, $e-1$, e, $e+1$; $\varlimsup\limits_{n\to\infty} x_n = e+1$, $\varliminf\limits_{n\to\infty} x_n = -e - \dfrac{1}{\sqrt{2}}$; (h) the limit points are 0, 1/2, 1; $\varlimsup\limits_{n\to\infty} x_n = 1$, $\varliminf\limits_{n\to\infty} x_n = 0$, (i) the limit points are 1, 2; $\varlimsup\limits_{n\to\infty} x_n = 2$, $\varliminf\limits_{n\to\infty} x_n = 1$, (j) the limit points are 0, 1; $\varlimsup\limits_{n\to\infty} x_n = 1$, $\varliminf\limits_{n\to\infty} x_n = 0$, (k) there are no limit points; $\varlimsup\limits_{n\to\infty} x_n = +\infty$, $\varliminf\limits_{n\to\infty} x_n = -\infty$. 36. It diverges.

● Prove that $\lim_{k\to\infty} x_{2k} \neq \lim_{k\to\infty} x_{2k+1}$. 37. ● (a) Use the estimation $\frac{1}{k!} \leqslant \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ for $k \geqslant 2$, (d) use the estimation $|x_{n+p} - x_n| = \left|\sum_{k=n+1}^{n+p} a_k q^k\right| \leqslant M \sum_{k=n+1}^{n+p} |q|^k$. 38. ● In Definition 1 of the fundamental sequence set $p=1$. 39. ● (b) Show that $|x_n - x_{2n}| \geqslant 1/2$ for $\varepsilon = 1/2$ and $\forall n$.

Chapter 3

2. ● Prove that $f(x)$ does not satisfy the definition of the limit of a function in the sense of Heine. 3. No, there is not. 4. ● To prove that there is no limit $\lim_{x\to a} f(x)$ for $|a| \neq 1$, use the negation of the definition of a limit of a function in the sense of Heine. 8. (a) 1, (b) 4/5, (c) $-1/2$, (d) 1, (e) m. 9. (a) 4/3, (b) -2, (c) 1/4. 10. (a) 1, (b) $5^{10}/3^{25}$, (c) 1, (d) $1/\sqrt[3]{2}$. 12. (a) No, there is not, (b) 1/2, (c) 1. 15. (a) $x = 0$ is a point of removable discontinuity, (b) $x = 0$ is a point of discontinuity of the second kind, (c) $x = k$ ($k \in \mathbb{Z}$) are points of discontinuity of the first kind, (d) at the points $x = 1$ and $x = -1$ the function is continuous and the other points are points of discontinuity of the second kind, (e) $x = -1$ is a point of discontinuity of the second kind, (f) $x = 0$ is a point of discontinuity of the first kind, (g) $x = 1$ is a point of discontinuity of the first kind, (h) $x = -1$ and $x = 3$ are points of discontinuity of the second kind, (i) $x = 1$ is a point of removable discontinuity, (j) $x = -1$ is a point of discontinuity of the first kind, (k) $x = -1$ is a point of discontinuity of the first kind. 17. $a^x = 1 + x \ln a + o(x)$, (b) $e^x = 1 + x + o(x)$, (c) $(1 + x)^a = 1 + ax + o(x)$, (d) $\sinh x = x + o(x)$, (e) $\tanh x = x + o(x)$, (f) $\cosh x = 1 + (1/2)x^2 + o(x^2)$. 19. (a) No, it does not, (b) yes, it does, (c) yes, it does. 20. (a) As $x \to 0$, the equality $o(x + x^2) = o(x^2)$ does not hold true. Indeed, the function $\alpha(x) = x\sqrt[3]{x}$, for example, is an infinitesimal of the order of smallness higher than that of $x + x^2$ as $x \to 0$ $\left(\text{since } \lim_{x\to 0} \frac{x\sqrt[3]{x}}{x+x^2} = 0\right)$, but $\lim_{x\to 0} \frac{x\sqrt[3]{x}}{x^2} = \infty$, i.e. $x\sqrt[3]{x} \neq o(x^2)$ as $x \to 0$, (b) no, it does not, (c) yes, it does, (d) no, it does not, (e) yes, it does. 23. (a) $25x + o(x)$, $25x + o(x)$, (b) $1 - 8x^4 + o(x^4)$, $1 - (1/2)x^2 + o(x^2)$, (c) $1 + 2x + o(x)$, $1 + \sqrt{x} + o(\sqrt{x})$, (d) $-x^2 + o(x^2)$, $x + o(x)$; (e) $(1/27)x + o(x)$, $-(1/27)\sqrt{x} + o(\sqrt{x})$; (f) $1 + x^3 \ln 2 + o(x^3)$, $1 + x^2 \ln 2 + o(x^2)$, (g) $-2x^2 + o(x^2)$, $-2x^2 + o(x^2)$, (h) $1 - (1/2)x + o(x)$, $1 + (1/2)x + o(x)$, (i) $\sqrt{x} + o(\sqrt{x})$, (j) $1 + (x + (1/2)|x|)\ln 5 + o(x)$, (k) $1 - (1/6)x + o(x)$, (l) $-(1/2)x^2 + o(x^2)$. 24. (a) $(x - 2)^2 + o((x - 2)^2)$, (b) $1 + \beta(2 - x) + o(2 - x)$, (c) $x - 2 + o(x - 2)$, (d) $1 - (1/2)\pi^2(x - 2)^2 + o((x - 2)^2)$, (e) $\pi(x^2 - 4) + o(x - 2)$, (f) $(2/35)(x - 2) +$

$o(x-2)$, (g) $4(1+\ln x)(x-2)+o(x-2)$. **25.** (a) $1/2+o(1)$, (b) $1/(3x)+o(1/x)$, (c) $5/2+o(1)$, (d) $1/\sqrt[3]{x^2}+o(1/\sqrt[3]{x^2})$, $1/\sqrt[3]{x^2}+o(1/\sqrt[3]{x^2})$, (e) $1-1/(2x^2)+o(1/x^2)$; (f) $1+(\ln 5)/x+o(1/x)$, (g) $-4/x^2+o(1/x^2)$, $4/x^2+o(1/x^2)$, (h) $1/\sqrt{x}+o(1/\sqrt{x})$. **26.** (a) $1/3+o(1)$, (b) $(1/n)\ln 14+o(1/n)$, (c) $1/\sqrt{n}+o(1/\sqrt{n})$, **27.** (a) 4, (b) $1/\sqrt{3}$, (c) $2/\pi$, (d) $\alpha/m-\beta/n$, (e) 1, (f) $-1/12$, (g) $1/a$, (h) $1/2$, $\sqrt{2/3}$, 1, (i) 0, (j) 0 if $a<b$, $+\infty$, if $a>b$, $e^{-1/b}$ if $a=b$, (k) 1, (l) 1, (m) $\sqrt[a]{b}$. **28.** (a) $\alpha/m+\beta/n$, (b) $-19/3$, (c) $1/324$, (d) $-4/9$, (e) 3, (f) $\ln a$, (g) e^{π^2}, (h) 0, (i) 0, (j) 0, (k) $e^{-x^2/2}$, (l) e^{-1}. **29.** (a) -2, (b) $a^a\ln(ea)$, (c) $2a/b$, (d) $a^x\ln^2 a$, (e) -2, (f) e^2, (g) $-\pi^2/4$, (h) e^{1-a}, (i) 0, (j) $\ln 2$. **30.** (a) 2, (b) $4/3$, (c) $7/4$, (d) $-\ln 2$, (e) $a^a\ln(a/e)$, (f) $\ln a$, (g) 2, (h) $45/91$, (i) $e^{-4\pi^2a^2}$, (j) $e^{2\alpha}$, (k) e, (l) $1/2$, (m) $1/2$.

Chapter 4

1. (a) $\Delta y=\arcsin\left(\frac{1}{2}+\Delta x\right)-\frac{\pi}{6}$, $-\frac{3}{2}\leqslant\Delta x\leqslant\frac{1}{2}$, (c) $\Delta y=\ln\left(1+\frac{\Delta x}{2}\right)$, $-2<\Delta x<+\infty$. **2.** (a) 1, (b) $2x_0$, (c) $y'(4)=\lim\limits_{\Delta x\to 0}\frac{\sqrt{4+\Delta x}-\sqrt{4}}{\Delta x}=\frac{1}{4}$, (d) 0, (e) $1/2$.
3. (a) $v_1>v_2$ for $0\leqslant t<1/2$, $v_1=v_2$ for $t=1/2$, $v_1<v_2$ for $t>1/2$; on $[0, 1]$ $v_{1\,\mathrm{av}}=v_{2\,\mathrm{av}}=1$, on $[1, 2]$ $v_{1\,\mathrm{av}}=1<3=v_{2\,\mathrm{av}}$, (b) $v_1=v_2=0$ for $t=0$, $v_1>v_2$ for $0<t<2/3$, $v_1=v_2=4/3$ for $t=2/3$, $v_1<v_2$ for $t>2/3$, on $[0, 1]$ $v_{1\,\mathrm{av}}=v_{2\,\mathrm{av}}=1$, on $[1, 2]$ $v_{1\,\mathrm{av}}=3<7=v_{2\,\mathrm{av}}$, (c) $v_1>v_2$ for $1\leqslant t<4$, $v_1=v_2=1/4$ for $t=4$, $v_1<v_2$ for $t>4$, on $[1, 4]$ $v_{1\,\mathrm{av}}=(1/3)\ln 4>1/3=v_{2\,\mathrm{av}}$, on $[1, 25]$ $v_{1\,\mathrm{av}}=(1/24)\ln 25<1/6=v_{2\,\mathrm{av}}$. **4.** (a) $y=x$, (b) $y=2x-1$, (c) $x=0$, (d) $y=\frac{1}{2}x+\frac{\pi}{4}-\frac{1}{2}$. **5.** (a) $\left(\frac{5\pi}{6}-\sqrt{3}, \sqrt{3}-\frac{\pi}{3}\right)$, (b) $\left(\frac{1}{e-1}, \frac{e}{e-1}\right)$, (c) $\left(\frac{6-\sqrt{3}\pi}{6(2-\sqrt{3})}, \frac{e-\sqrt{3}\pi}{6(2-\sqrt{3})}\right)$. **6.** $y=\frac{1}{2}x+\frac{1}{2}$, $y=\frac{1}{4}x+1$.
7. (a) $f'(+0)=1$, $f'(-0)=-1$, (b) $f'(1+0)=f'(1-0)=f'(1)=1$, (c) $f'(+0)=f'(-0)=f'(0)=0$, (d) $f'(+0)=1$, $f'(-0)=-1$, (e) $f'(+0)=f'(-0)=f'(0)=0$, (f) $f'(\pi/2+0)=f'(\pi/2-0)=f'(\pi/2)=0$, (g) $f'(1+0)=e$, $f'(1-0)=-e$.
8. (a) $2x$, (b) $\frac{1}{2\sqrt{x}}$ $(x>0)$, (c) $-\frac{1}{x^2}$ $(x\neq 0)$, (3) $\frac{4}{3\sqrt[3]{x}}+\frac{3}{2x\sqrt{x}}$ $(x>0)$, (e) $\frac{1}{x}\,\frac{\ln 108}{\ln 2\ln 3}$ $(x>0)$, $y'(1)=\frac{\ln 108}{\ln 2\ln 3}$, (f) $\left(2^x-\frac{1}{2^x}\right)\ln 2$, (g) $\cos x+\sin x$, $y'(0)=1$, $y'(\pi/4)=\sqrt{2}$,

(h) $1/(\sin^2 x \cos^2 x)$ $(x \neq \pi n/2,\ n \in \mathbf{Z})$, (i) 0 $(\arcsin x + \arccos x = \pi/2 = \text{const})$, (j) 0 $(\arctan x + \operatorname{arccot} x = \pi/2 = \text{const})$. 9. ● Represent $[u(x)]^{v(x)}$ in the form $e^{v(x) \ln u(x)}$ and use the formula for the derivative of a complex function. 10. (a) $\dfrac{ad-bc}{(cx+d)^2}$, $\dfrac{x}{\sqrt{x^2-a^2}}$ $(x < -a,\ x > a)$, $\dfrac{5x^2+3}{3\sqrt[3]{(x^2+1)^2}}$, (b) $\dfrac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \times \left[1 + \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)\right]$ $(x > 0)$, $-\sin x \sin \times (2\cos x) - \cos x \sin(2\sin x)$, (c) $\cos[\sin(\sin x)]\cos(\sin x)\cos x$, $\dfrac{4\tan x\,(1+\tan^2 x)}{(1-\tan^2 x)^2}$ $\left(x \neq \dfrac{\pi n}{4},\ n \in \mathbf{Z}\right)$; $2^{\cos x + \tan x} \ln 2 \times \left(-\sin x + \dfrac{1}{\cos^2 x}\right)$ $\left(x \neq \dfrac{\pi}{2} + \pi n,\ n \in \mathbf{Z}\right)$, (d) $e^x(\sin x + \cos x)$, $2e^{x^2}(x\cos 2x - \sin 2x)$, $e^x\left[e^{e^x} + x^{e^x}\left(\ln x + \dfrac{1}{x}\right)\right]$ $(x > 0)$, (e) $x^x(\ln x + 1)$ $(x > 0)$, $\dfrac{1}{x \ln x\,[\ln(\ln x)]}$ $(x > e)$, $\dfrac{1}{x^2-a^2}$ $(x < -a,\ x > a)$, (f) $\dfrac{1}{x}$ $(x \neq 0)$, $\dfrac{1}{\sqrt{x^2 \pm a^2}}$, $\cot x$ $(2\pi n < x < \pi + 2\pi n,\ n \in \mathbf{Z})$, (g) $\dfrac{1}{x}\cos(\ln x)$ $(x > 0)$, $\dfrac{1}{\sqrt{a^2-x^2}}$ $(-a < x < a)$, $\dfrac{1}{x^2+a^2}$, (h) $\dfrac{1}{1+x^2}$ $(x \neq 1)$, $\arctan\dfrac{1+x}{1-x} - \arctan x = \begin{cases} \pi/4 & \text{for } x < 1, \\ -3\pi/4 & \text{for } x > 1, \end{cases}$ (i) $\dfrac{1}{|x|\sqrt{x^2-1}}$ $(x < -1,\ x > 1)$, $\operatorname{sgn}\cos x$ $(x \neq \pi n/2,\ n \in \mathbf{Z})$, 1 $\left(x \neq \dfrac{\pi}{2} + \pi n,\ n \in \mathbf{Z}\right)$, (j) 1 $(-1 \leqslant x \leqslant 1)$, 1, (k) $\dfrac{a^2+b^2}{(x+a)(x^2+b^2)}$ $(x > -a)$, $\sqrt{a^2-x^2}$ $(-a < x < a)$, (l) $-\dfrac{\arccos x}{x^2}$ $(0 < |x| < 1)$, $\dfrac{x \ln x}{\sqrt{(x^2-1)^3}}$ $(x > 1)$, (m) $\dfrac{x \arcsin x}{\sqrt{(1-x^2)^3}}$ $(-1 < x < 1)$, $\dfrac{e^x}{\sqrt{1+e^{2x}}}$, (n) $\dfrac{1}{2(1+x^2)}$, $(\sin x)^{\cos x - 1}(\cos^2 x - \sin^2 x \ln \sin x)$ $(2\pi n < x < \pi + 2\pi n,\ n \in \mathbf{Z})$; $\dfrac{\cosh(\tan x)}{\cos^2 x}$ $\left(x \neq \dfrac{\pi}{2} + \pi n,\ n \in \mathbf{Z}\right)$, $-\dfrac{\sin x}{\cosh^2(\cos x)}$, (o) $\coth x$ $(x > 0)$, $\tanh x/\ln 10$; $1/\cosh 2x$, $-1/\sinh x$ $(x > 0)$.
11. (a) $\dfrac{\varphi\varphi' + \psi\psi'}{\sqrt{\varphi^2+\psi^2}}$ $(\varphi^2 + \psi^2 \neq 0)$, (b) $\dfrac{\psi'\varphi \ln\varphi - \varphi'\psi \ln\psi}{\varphi\psi \ln^2\varphi}$,

(c) $2xf'(x^2)-\frac{2}{x^3}f'(x^{-2})$ $(x \neq 0)$, (d) $f'(f(x))\,f'(x)$. 14. (a) No, we cannot, $(uv)'|_{x=0}=0$, (b) yes, we can, $(uv)'|_{x=1}=1/2$, (c) yes, we can, $(uv)'|_{x=1}=\cos 1$, (d) no, we cannot, $(uv)'|_{x=0}=0$, (e) no, we cannot, $(uv)'|_{x=0}=0$. 15. I. (a) Yes, it is, (b) no, it isn't. II. (a) No, it isn't, (b) no, it isn't. 16. No, it isn't. 17. ● Consider $(x+x^2+\ldots+x^n)'$. 18. (a) $-\infty<t<\infty$, $f'(x)=2t/1\,|_{t=x}=2x$, $f(x)=x^2$ $(-\infty<x<\infty)$, (b) $0\leqslant t\leqslant \pi/2$, $f'(x)=(\sin^2 t)'/(\cos^2 t)'\Big|_{\substack{t=\arccos\sqrt{x}\\ t\neq 0,\ t\neq\pi/2}}=-1$ $(0<x<1)$, $f(x)=1-x$ $(0\leqslant x\leqslant 1)$, (c) $0\leqslant t\leqslant \pi$, $f'(x)=\frac{b\cos t}{-a\sin t}\Big|_{\substack{t=\arccos(x/a),\\ t\neq 0,\ t\neq\pi}}=-\frac{b}{a}\frac{\cos t}{\sqrt{1-\cos^2 t}}\Big|_{\substack{\cos t=x/a,\\ t\neq 0,\ t\neq\pi}}=-\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}}$ $(-a<x<a)$, $f(x)=\frac{b}{a}\sqrt{a^2-x^2}$ $(-a\leqslant x\leqslant a)$, a tangent: $x=a$, a normal: $y=0$, (d) $0\leqslant t<\infty$, $f'(x)=\frac{b\cosh t}{a\sinh t}\Big|_{0<t<\infty}=\frac{b}{a}\frac{x}{\sqrt{x^2-a^2}}$ $(x>a)$, $f(x)=\frac{b}{a}\sqrt{x^2-a^2}$ $(x\geqslant a)$, a tangent: $x=a$, a normal: $y=0$, (f) $-\infty<t<\infty$, $f'(x)=\frac{2e^{2t}}{e^t}\Big|_{t=\ln x}=2x$, $f(x)=x^2$ $(0<x<\infty)$. 22. (a) $|\mathbf{v}|=\sqrt{18}$, $\cos X=\cos Y=1/\sqrt{18}$, $\cos Z\ 4/\sqrt{18}$, (b) $\mathbf{v}=\sqrt{R^2+h^2}$, $\cos X=-R\sqrt{R^2+h^2}$, $\cos Y=0$, $\cos Z=h/\sqrt{R^2+h^2}$, (c) $|\mathbf{v}|=\sqrt{14}$, $\cos X=1/\sqrt{14}$, $\cos Y=2/\sqrt{14}$, $\cos Z=3/\sqrt{14}$, (d) $|\mathbf{v}|=2.9$, $\cos X=4/29$, $\cos Y=25/29$, $\cos Z=10\sqrt{2}/29$. 23. (a) $\Delta y=\Delta x+\alpha(\Delta x)\,\Delta x$, where $\alpha(\Delta x)=\begin{cases}\frac{e^{\Delta x}-1-\Delta x}{\Delta x} & \text{for } \Delta x\neq 0,\\ 0 & \text{for } \Delta x=0,\end{cases}$ (b) $\Delta y=\alpha(\Delta x)\,\Delta x$, where $\alpha(\Delta x)=\begin{cases}\frac{\sin(\pi/2+\Delta x)-1}{\Delta x} & \text{for } \Delta x\neq 0,\\ 0 & \text{for } \Delta x=0,\end{cases}$ (c) $\Delta y=\Delta x+\alpha(\Delta x)\,\Delta x$, where $\alpha(\Delta x)=\begin{cases}\frac{\arctan\Delta x-\Delta x}{\Delta x} & \text{for } \Delta x\neq 0,\\ 0 & \text{for } \Delta x=0.\end{cases}$
24. $\Delta y=\Delta x+2(\Delta x)^2+(\Delta x)^3$, $dy=\Delta x$, (a) $\Delta y=0.010201$, $dy=0.01$, (b) $\Delta y=0.121$, $dy=0.1$, (c) $\Delta y=4$, $dy=1$, (d) $\Delta y=48$, $dy=3$. 25. $\Delta s=5\Delta t+2\Delta t^2$, $ds=5\Delta t$, (a) $\Delta s=0.52$, $ds=0.5$, (b) $\Delta s=1.08$, $ds=1$, (c) $\Delta s=7$, $ds=5$. 26. (a) $\frac{dx}{2\sqrt{x}}$ $(x>0)$, (b) $-\frac{dx}{x^2}$ $(x\neq 0)$, (c) $\frac{dx}{\sqrt{x^2+1}}$, (d) $\frac{dx}{x^2-1}$ $(x\neq\pm 1)$, (e) $\frac{dx}{\sqrt{a^2-x}}$

$(-a < x < a)$, (f) $\dfrac{dx}{x^2+a^2}$, (g) $(1+2x)\,e^{2x}\,dx$, (h) $x\cos x\,dx$.

27. (a) $dy\,|_{x=0} = dx$, $dy\,|_{x=1} = dx$, (b) $dy\,|_{x=0} = dx$, $dy\,|_{x=1} = \dfrac{1}{2}\,dx$, (c) $dy\,|_{x=0} = dx$, $dy\,|_{x=1} = e\,dx$, (d) $dy\,|_{x=0} = \dfrac{\pi}{2}\,dx$, $dy\,|_{x=1} = 0$, (e) $dy\,|_{x=0} = 0$, $dy\,|_{x=1} = -\dfrac{\pi}{2}\,dx$.

29. Equalities (b) and (c). 30. (a) -0.8747, (b) 0.5121 rad. or $29°20'$, (c) 1.04, (d) 1.0033, (e) 0.83 rad. or $47°33'$, (f) 1.2.

31. (a) 2.08, (b) 3.9961, (c) 2.0045. 32. (a) $(12x-8x^3)\,e^{-x^2}$, (b) $-a^{10}\sin ax$, (c) k^4e^{kx}, (d) $12xf''(x^2)+8x^3f'''(x^2)$, (e) $e^{x}f'(e^x)+e^{2x}f''(e^x)$, (f) $\varphi'''(x)\,f'(\varphi(x))+3\varphi'(x)\,\varphi''(x)\,f''(\varphi(x))+\varphi'^3(x)\,f'''(\varphi(x))$, (g) $-\dfrac{1\cdot3\cdot5\ldots17}{2^{10}x^9\sqrt{x}}$ $(x>0)$, (h) $\dfrac{720}{(x-1)^7}$ $(x\neq1)$, (i) $2^{20}\,(x^2\sin 2x-20x\cos 2x-95\sin 2x)$, (j) $5^{14}\,(5x^3-126x)\times\sin 5x-3\cdot5^{13}\,(75x^2-182)\cos 5x$, (k) $-\dfrac{2\cdot8!}{(x+1)^9}$ $(x\neq-1)$, (l) $\dfrac{1}{2}\cdot30!\left[\dfrac{1}{(x-1)^{31}}+\dfrac{1}{(x+1)^{31}}\right]$ $(x\neq\pm1)$, (m) $5^{10}\,(5x+11)\,e^{5x}$, (n) $-\dfrac{9!}{x^{10}}$ $(x>0)$. 33. (a) $\dfrac{(-1)^{n-1}\,(2n-3)\,!!\left(\dfrac{a}{2}\right)^n}{(ax+b)^{\frac{2n-1}{2}}}$ $(ax+b>0)$, (b) $\dfrac{(-1)^{n-1}\,(ad-bc)\,c^{n-1}}{(cx+d)^{n+1}}$ $(cx+d\neq0)$, (c) $-2^{n-1}\cos\left(2x+n\dfrac{\pi}{2}\right)$, (d) $2^{n-1}\cos\left(2x+n\dfrac{\pi}{2}\right)$, (e) $\dfrac{3}{4}\sin\left(x+n\dfrac{\pi}{2}\right)-\dfrac{3^n}{4}\sin\left(3x+n\dfrac{\pi}{2}\right)$, (f) $\dfrac{3}{4}\cos\left(x+n\dfrac{\pi}{2}\right)+\dfrac{3^n}{4}\cos\left(3x+n\dfrac{\pi}{2}\right)$, (g) $\dfrac{1}{2}(\alpha-\beta)\cos\left[(\alpha-\beta)\,x+n\dfrac{\pi}{2}\right]-\dfrac{1}{2}(\alpha+\beta)^n\cos\left[(\alpha+\beta)\,x+n\dfrac{\pi}{2}\right]$, (h) $\dfrac{1}{2}(\alpha-\beta)^n\times\cos\left[(\alpha-\beta)\,x+n\dfrac{\pi}{2}\right]+\dfrac{1}{2}(\alpha+\beta)^n\cos\left[(\alpha+\beta)\,x+n\dfrac{\pi}{2}\right]$, (i) $a^{n-1}\left[ax\sin\left(ax+n\dfrac{\pi}{2}\right)+n\sin\left(ax+(n-1)\dfrac{\pi}{2}\right)\right]$, (j) $a^{n-2}\left[a^2x^2\cos\left(ax+n\dfrac{\pi}{2}\right)+2nax\cos\left(ax+(n-1)\dfrac{\pi}{2}\right)+n(n-1)\cos\left(ax+(n-2)\dfrac{\pi}{2}\right)\right]$, (k) $k^{n-2}e^{kx}\,[(ax^2+bx+c)\,k^2+(2ax+b)\,nk+n(n-1)\,a]$; (l) $(-1)^{n-1}a^n\,(n-1)\,!\left[\dfrac{1}{(ax+b)^n}-\dfrac{1}{(ax-b^n)}\right]\left(\dfrac{ax+b}{ax-b}>0\right)$, (m) $x\cosh x+n\sinh x$ if n is odd;

$x \sinh x + n \cosh x$ if n is even, (n) $x^2 \sinh x + 2nx \cosh x + n(n-1) \sinh x$ if n is odd, $x^2 \cosh x + 2nx \sinh x + n(n-1) \cosh x$ if n is even, (o) $a_0 n!$. **34.** (a) $f''(x) = 2$, $f'''(x) = 0$, (b) $f''(x) = f'''(x) = 0$, (c) $f''(x) = -\dfrac{ab}{(a^2-x^2)^{3/2}}$, $f'''(x) = -\dfrac{3abx}{(a^2-x^2)^{5/2}}$, (d) $f''(x) = -\dfrac{ab}{(x^2-a^2)^{3/2}}$, $f'''(x) = \dfrac{3abx}{(x^2-a^2)^{5/2}}$, (e) $f''(x) = -\dfrac{1}{4a \sin^4 (t/2)}$, $f'''(x) = \dfrac{\cos (t/2)}{4a^2 \sin^7 (t/2)}$, where $t = \varphi^{-1}(x)$ is the inverse of the function $x = a(t - \sin t)$ $(t \neq 2\pi n,\ n \in \mathbf{Z})$, (f) $f''(x) = 2$, $f'''(x) = 0$. **35.** $(f^{-1}(y))' = \dfrac{1}{f'(x)}$, $(f^{-1}(y))'' = -\dfrac{f''(x)}{f'^3(x)}$, $(f^{-1}(y))''' = \dfrac{3f''^2(x) - f'(x) f'''(x)}{f'^5(x)}$. **36.** (a) $|\mathbf{r}''(2)| = 2$, $\cos X = \cos Y = 0$, $\cos Z = 1$, (b) $|\mathbf{r}''(\pi)| = 1$, $\cos X = 1$, $\cos Y = \cos Z = 0$, (c) $|\mathbf{r}''(1)| = 2\sqrt{10}$, $\cos X = 0$, $\cos Y = 1/\sqrt{10}$, $\cos Z = 3/\sqrt{10}$, (d) $|\mathbf{r}''|(2.5) = \sqrt{641}/25$, $\cos X = -4/\sqrt{641}$, $\cos Y = 25/\sqrt{641}$, $\cos Z = 0$. **37.** (a) $6 \mathrm{d}x^3$, (b) $\dfrac{-15 \mathrm{d}x^4}{16(x-1)^{7/2}}$ $(x > 1)$, (c) $-\dfrac{6 \mathrm{d}x^5}{x^4}$ $(x > 0)$, (d) $(10 \cos x - x \sin x) \mathrm{d}x^{10}$. **38.** (a) $\cosh x \, \mathrm{d}x^n$ if n is odd, $\sinh x \, \mathrm{d}x^n$ if n is even, (b) $a^n \sinh (ax) \mathrm{d}x^n$ if n is odd, $a^n \cosh (ax) \mathrm{d}x^n$ if n is even, (c) $(-1)^{n-1} 2 \cdot (n-3)! \dfrac{\mathrm{d}x^n}{x^{n-2}}$ $(x > 0,\ n \geqslant 3)$. **40.** $\varphi(x_0) \cdot n!$.

Chapter 5

1. $27x - 9x^3 + 9x^5/5 - x^7/7$. **2.** $4(x^2+7)/(7\sqrt[4]{x})$. **3.** $\ln |x| - 1/(4x^4)$. **4.** $x + 2 \ln \left| \dfrac{x-1}{x+1} \right|$. **5.** $-\dfrac{2}{\ln 5} \left(\dfrac{1}{5} \right)^x + \dfrac{1}{5 \ln 2} \left(\dfrac{1}{2} \right)^x$. **6.** $x - \tanh x$. **7.** $\arcsin x + \ln (x + \sqrt{1+x^2})$. **8.** $\dfrac{4^x}{\ln 4} + 2 \dfrac{6^x}{\ln 6} + \dfrac{9^x}{\ln 9}$. **9.** $(-\cos x + \sin x) \operatorname{sgn} (\cos x + \sin x)$. **10.** $(1/22)(2x-3)^{11}$. **11.** $-(2/5)\sqrt{2-5x}$. **12.** $(1/\sqrt{6}) \arctan (x\sqrt{3/2})$. **13.** $(1/\sqrt{3}) \times \ln |x\sqrt{3} + \sqrt{3x^2-2}|$. **14.** $-\dfrac{1}{2} \cot \left(2x + \dfrac{\pi}{4} \right)$. **15.** $\tan (x/2)$. **16.** $\tan \left(\dfrac{x}{2} - \dfrac{\pi}{4} \right)$. **17.** $(1/4)(1+x^3)^{4/3}$. **18.** $(1/4) \arctan (x^2/2)$. **19.** $2 \arctan \sqrt{x}$. ● Use the fact that $\dfrac{\mathrm{d}x}{\sqrt{x}} = 2\mathrm{d}(\sqrt{x})$
20. $-\arcsin (1/|x|)$. **21.** $2 \operatorname{sgn} x \ln (\sqrt{|x|} + \sqrt{|1+x|})$, $x(1+x) > 0$. **22.** $-(1/2) e^{-x^2}$. **23.** $(1/3) \ln^3 x$. **24.** $(1/6) \sin^6 x$. **25.** $(3/2) \times \sqrt[3]{1 - \sin 2x}$. **26.** $-(1/\sqrt{2}) \ln |\sqrt{2} \cos x + \sqrt{\cos 2x}|$. **27.** $\ln |\tan \times$

$(x/2)|$. 28. $\ln|\tanh(x/2)|$. 29. $0.5(\arctan x)^2$. 30. $\frac{1}{4}\ln^2\frac{1+x}{1-x}$.

31. $-\frac{1+55x^2}{6600}(1-5x^2)^{11}$. 32. $-(1/15)(8+4x^2+3x^4)\sqrt{1-x^2}$.

33. $\left(\frac{2}{3}-\frac{4}{7}\sin^2 x+\frac{2}{11}\sin^4 x\right)\sqrt{\sin^3 x}$. 34. $-x-2e^{-x/2}+2\ln(1+e^{x/2})$. 35. $x-2\ln(1+\sqrt{1+e^x})$. 36. $(\arctan\sqrt{x})^2$. 37. $0.5[x\sqrt{a^2-x^2}+a^2\arcsin(x/a)]$. 38. $-a\sqrt{1-x^2/a^2}+a\arcsin(x/a)$. 39. $0.5\times[x\sqrt{a^2+x^2}-a^2\ln(x+\sqrt{a^2+x^2})]$. 40. $\sqrt{x^2-a^2}-2a\ln(\sqrt{x-a}+\sqrt{x+a})$ if $x>a$, $-\sqrt{x^2-a^2}+2a\ln(\sqrt{-x+a}+\sqrt{-x-a})$ if $x<-a$. 41. $\ln|x+\sqrt{x^2+a^2}|$. 42. $\ln|x+\sqrt{x^2-a^2}|$.

43. $x(\ln x-1)$. 44. $\frac{2}{3}x^{3/2}\left(\ln^2 x-\frac{4}{3}\ln x+\frac{8}{9}\right)$. 45. $-0.5(x^2+1)e^{-x^2}$. 46. $\frac{1-2x^2}{4}\cos 2x+\frac{x}{2}\sin 2x$. 47. $x\arcsin x+\sqrt{1-x^2}$.

48. $-\frac{\arcsin x}{x}-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|$. 49. $\ln\tan(x/2)-\cos x\ln\tan x$.

50. $0.5[(1+x^2)(2\arctan x)^2-2x\arctan x+\ln(1+x^2)]$. 51. $\sqrt{1+x^2}\times\ln(x+\sqrt{1+x^2})-x$. 52. $0.5(x\sqrt{x^2\pm a^2}\pm a^2\ln|x+\sqrt{x^2\pm a^2}|)$.

53. $0.5x[\sin(\ln x)+\cos(\ln x)]$. 54. $\frac{a\sin bx-b\cos bx}{a^2+b^2}e^{ax}$.

55. $(1/8)e^{2x}(2-\sin 2x-\cos 2x)$. 56. $\ln|x-2|+\ln|x+5|$.

57. $x+(1/6)\ln|x|-(9/2)\ln|x-2|+(28/3)\ln|x-3|$.

58. $-\frac{5x-6}{x^2-3x+2}+4\ln\left|\frac{x-1}{x-2}\right|$. 59. $\arctan x+\frac{5}{6}\ln\frac{x^2+1}{x^2+4}$.

60. $-\frac{1}{x-2}-\arctan(x-2)$. 61. $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$.

62. $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x$. 63. $\frac{1}{4\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\arctan\frac{x\sqrt{2}}{1-x^2}$. 64. $\frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\arctan\frac{x^2-1}{x\sqrt{3}}$.

65. $\frac{2}{5}\ln\frac{x^2+2x+2}{x^2+x+\frac{1}{2}}+\frac{8}{5}\arctan(x+1)-\frac{2}{5}\arctan(2x+1)$.

66. $\frac{1}{4\sqrt{3}}\ln\frac{1+x\sqrt{3}+x^2}{1-x\sqrt{3}+x^2}+\frac{1}{2}\arctan x+\frac{1}{6}\arctan x^3$.

67. $-\frac{1}{96(x-1)^{96}}-\frac{3}{97(x-1)^{97}}-\frac{3}{98(x-1)^{98}}-\frac{1}{99(x-1)^{99}}$.

68. $\frac{1}{12}\ln\frac{(x^2+1)^2}{x^4-x^2+1}+\frac{1}{3}\arctan x^3+\frac{1}{2\sqrt{3}}\arctan\frac{2x^2-1}{\sqrt{3}}$.

69. $\frac{x^4}{4}+\frac{1}{4}\ln\frac{x^4+1}{(x^4+2)^4}$. 70. $\frac{1}{7}\ln\frac{|x|^7}{(1+x^7)^2}$. 71. $\frac{1}{\sqrt{3}}\arctan\frac{x^2-1}{x\sqrt{3}}$.

72. $\frac{1}{4\sqrt{2}}\ln\frac{x^4-x^2\sqrt{2}+1}{x^4+x^2\sqrt{2}+1}$. 73. $\arctan x+\frac{1}{3}\arctan x^3$.

74. $\frac{3}{4}\ln\frac{x\sqrt[3]{x}}{(1+\sqrt[6]{x})^2(1-\sqrt[6]{x}+2\sqrt[6]{x})^3}-\frac{3}{2\sqrt{7}}\arctan\frac{4\sqrt[6]{x}-1}{\sqrt{7}}$.

75. $0.5\,(x^2-x\sqrt{x^2-1}+\ln|x+\sqrt{x^2-1}|)$. 76. $-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}$.

77. $\frac{2x-3}{4}\sqrt{x^2+x+1}-\frac{1}{8}\ln\left(\frac{1}{2}+x+\sqrt{x^2+x+1}\right)$.

78. $-\ln\left|\frac{2-x+2\sqrt{x^2+x+1}}{x+1}\right|$. 79. $R+\ln(x+1+R)-\sqrt{2}\times$ $\ln\left|\frac{x+2+\sqrt{2R}}{x}\right|$, where $R=\sqrt{x^2+x+2}$. 80. $-\frac{19+5x+2x^2}{6}\times$ $\sqrt{1+2x-x^2}-4\arcsin\frac{1-x}{\sqrt{2}}$. 81. $\left(\frac{63}{256}x-\frac{21}{128}x^3+\frac{21}{160}x^5-\frac{9}{80}x^7+\frac{x^9}{10}\right)\sqrt{1+x^2}-\frac{63}{256}\ln(x+\sqrt{1+x^2})$. 82. $-\frac{1}{2x^2}\sqrt{x^2+1}+$ $\frac{1}{2}\ln\frac{1+\sqrt{x^2+1}}{|x|}$. 83. $\frac{2x^2+1}{3x^3}\sqrt{x^2-1}$. 84. $\arcsin\frac{x-1}{\sqrt{3}}-$ $\frac{\sqrt{2}}{3}\arctan\frac{\sqrt{2+2x-x^2}}{(1-x)\sqrt{2}}-\frac{1}{\sqrt{6}}\ln\frac{\sqrt{6}+\sqrt{2+2x-x^2}}{\sqrt{6}-\sqrt{2+2x-x^2}}$.

85. $\frac{x}{2\sqrt{1+x^2}}+\frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{1+x^2}+x\sqrt{2}}{\sqrt{1+x^2}-x\sqrt{2}}\right|$. 86. $\frac{1}{2}\times$ $\arcsin\frac{x-3}{|x-1|\sqrt{5}}-\frac{1}{2}\ln\left|\frac{3x+1-2\sqrt{x^2-x-1}}{x+1}\right|$.

87. $\frac{2(x-1)}{3\sqrt{x^2+x+1}}$. 88. $\frac{3}{2(2z+1)}+\frac{1}{2}\ln\frac{z^4}{|2z+1|^3}$, where $z=x+\sqrt{x^2+x+1}$. 89. $\ln\left|\frac{z-1}{z}\right|-2\arctan z$, where $z=\frac{1+\sqrt{1-2x+x^2}}{x}$. 90. $\frac{2(3-4z)}{5(1-z-z^2)}+\frac{2}{5\sqrt{5}}\ln\left|\frac{\sqrt{5}+1+2z}{\sqrt{5}-1-2z}\right|$, where $z=-x+\sqrt{x(1+x)}$. 91. $-\frac{1}{\sqrt{2}}\arcsin\frac{x\sqrt{2}}{x^2+1}$. ● Set $t=x+\frac{1}{x}$. 92. $-\frac{1}{\sqrt{2}}\ln\left|\frac{x\sqrt{2}+\sqrt{x^4+1}}{x^2-1}\right|$. ● Set $t=x-\frac{1}{x}$.

93. $(5/16)x-(1/4)\sin 2x+(3/64)\sin 4x+(1/48)\sin^3 2x$. 94. $(1/16)x-(1/64)\sin 4x+(1/48)\sin^3 2x$. 95. $1/(3\cos^3 x)-1/\cos x$. 96. $(1/4)\times\tan^4 x-(1/2)\tan^2 x-\ln|\cos x|$. 97. $-2\sqrt{\cot x}+(2/3)\sqrt{\tan^3 x}$.

98. $\frac{1}{2\sqrt{2}}\ln\frac{z^2+z\sqrt{2}+1}{z^2-z\sqrt{2}+1}-\frac{1}{\sqrt{2}}\arctan\frac{z\sqrt{2}}{z^2-1}$, where $z=\tan x$.

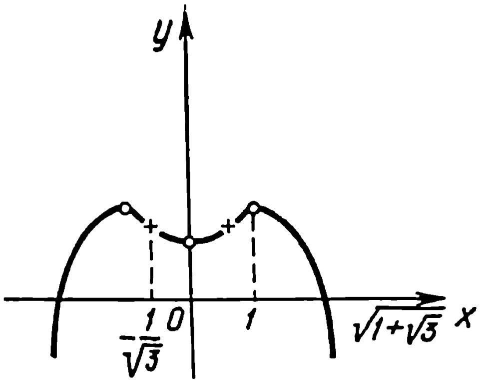
99. $(1/4)x+(1/8)\sin 2x+(1/16)\sin 4x+(1/24)\sin 6x$. 100. $-(3/16)\times\cos 2x+(3/64)\cos 4x+(1/48)\cos 6x-(3/128)\cos 8x+(1/192)\cos 12x$.

101. $\frac{1}{\sqrt{5}}\arctan\frac{3\tan\frac{x}{2}+1}{\sqrt{5}}$. 102. (a) $\frac{2}{\sqrt{1-\varepsilon^2}}\times$

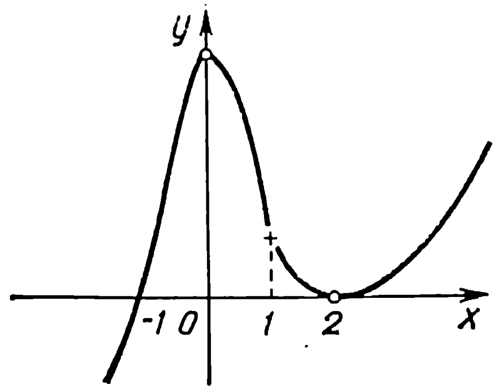
$\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{x}{2}\right)$, (b) $\frac{1}{\sqrt{\varepsilon^2-1}}\ln\frac{\varepsilon+\cos x+\sqrt{\varepsilon^2-1}\sin x}{1+\varepsilon\cos x}$.
103. $x-\frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)$. 104. $-\frac{1}{6}\ln\frac{(\sin x+\cos x)^2}{1-\sin x\cos x}-\frac{1}{\sqrt{3}}\arctan\left(\frac{2\cos x-\sin x}{\sqrt{3}\sin x}\right)$. 105. $\arctan\left(\frac{1}{2}\tan 2x\right)$.

Chapter 6

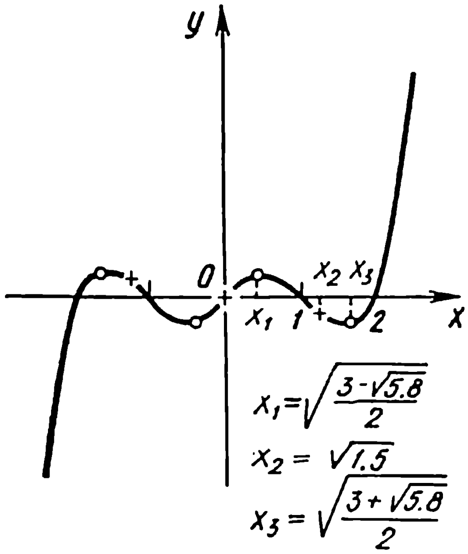
2. (a) Yes, it is, (b) no, it isn't, (c) no, it isn't, (d) no, it isn't, (e) yes, it is. 4. (a) No, it does not, (b) yes, it does. 8. (a) $\inf\limits_{(0,+\infty)} f(x)=0$ is not attained, $\sup\limits_{(0,+\infty)} f(x)=1=f(1)$, (b) $\inf\limits_{[-5,10]} f(x)=0=f(0)$, $\sup\limits_{[-5,10]} f(x)=100=f(10)$, (c) $\inf\limits_{(-\infty,+\infty)} f(x)=0$, $\sup\limits_{(-\infty,+\infty)} f(x)=\pi/2$ are not attained, (d) $\inf\limits_{[0,\pi]} f(x)=-1=f(\pi)$, $\sup\limits_{[0,\pi]} f(x)=\sqrt{2}=f(\pi/4)$, (e) $\inf\limits_{(0,1)} f(x)=0$, $\sup\limits_{(0,1)} f(x)=1/2$ are not attained. 13. (a) 4, (b) 2, (c) $8/(3\pi)$, (d) $2/\pi$. 17. (a) $\delta(\varepsilon)=\varepsilon/|k|$, (b) $\delta(\varepsilon)=\varepsilon/75$, (c) $\delta(\varepsilon)=\varepsilon$, (d) $\delta(\varepsilon)=e^{-10}\varepsilon$. 18. (a) It is uniformly continuous on (1, 2) but not uniformly continuous on (0, 1), (b) it is uniformly continuous on (0.01, 1) but not uniformly continuous on (0, 1), (c) it is uniformly continuous, (d) it is uniformly continuous, (e) it is uniformly continuous. 22. $f(x)=\arcsin x$ on $(-1, 1)$. 23. The uniform continuity of $f(x)$ on (a, b) implies that Cauchy's condition for the existence of a limit of a function is fulfilled at the points a and b. Therefore we can extend the definition of the function $f(x)$ at the points a and b so that it will become continuous on $[a, b]$ and, hence, bounded on $[a, b]$. 27. It decreases on $\left(-\infty, -\frac{b}{2a}\right)$, increases on $\left(-\frac{b}{2a}, +\infty\right)$, (b) it increases on $(-\infty, +\infty)$, (c) it increases on $(-1, 1)$, decreases on $(-\infty, -1)$ and $(1, +\infty)$, (d) it increases on $(-\infty, +\infty)$, (e) it increases on $(2\pi n-2\pi/3, 2\pi n+2\pi/3)$, decreases on $(2\pi n+2\pi/3, 2\pi n+4\pi/3)$, $n\in Z$, (f) it increases on $\left(\frac{1}{2n+1.5}, \frac{1}{2n+0.5}\right)$, $n\in Z$, decreases on $(-\infty, -2)$, $\left(\frac{1}{2n+0.5}, \frac{1}{2n-0.5}\right)$, $n\in Z$, $n\neq 0$ and $(2, +\infty)$, (g) it increases on $(0, 2/\ln 2)$, decreases on $(-\infty, 0)$ and $(2/\ln 2, +\infty)$, (h) it increases on $(0, n)$, decreases on $(n, +\infty)$. 29. Use the method employed in Example 3 in 6.3. 32. $c=1/2$ or $\sqrt{2}$. 34. ● Use the method employed in Example 5 in 6.3. 36. ● Use Theorem 6 and the result of Exercise 23. 37. No, it is not. 38. Consider the function $g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}\times(x-a)$. It is continuous on $[a, b]$ and differentiable in (a, b), with $g(a)=g(b)=0$. Since $f(x)$ is not a linear function, it follows that $g(x)\not\equiv 0$, and consequently, $g'(x)\not\equiv 0$ in (a, b). Hence $\exists c_1$,



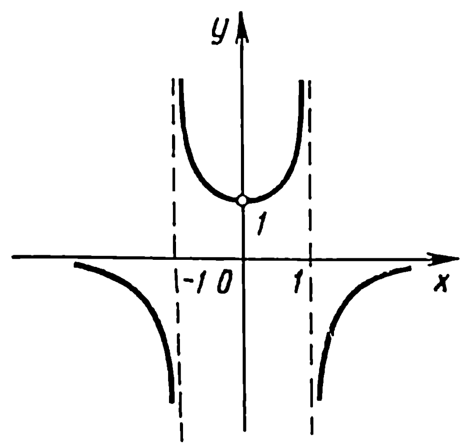
To Ex. 1



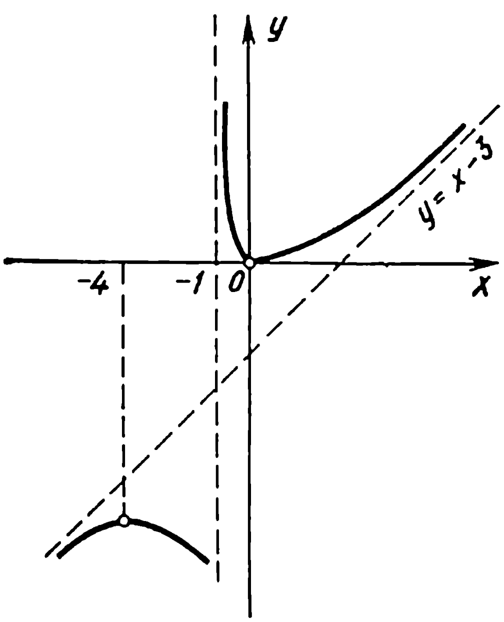
To Ex. 2



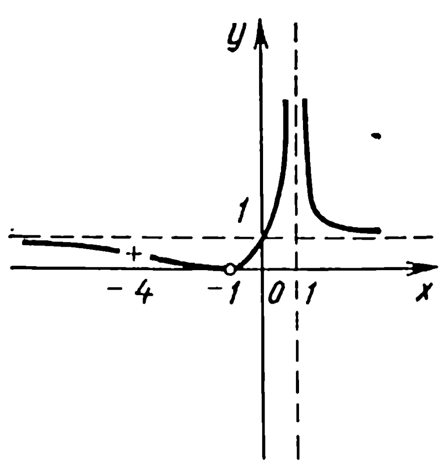
To Ex. 3



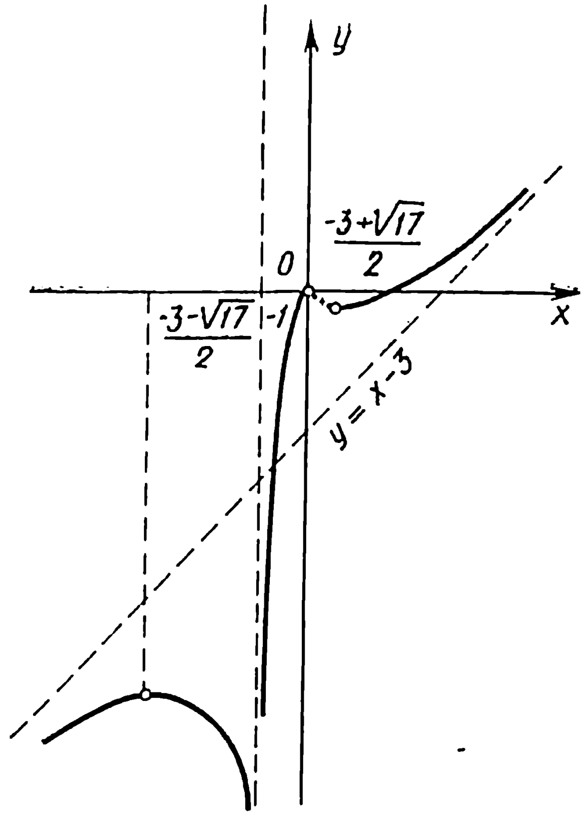
To Ex. 4



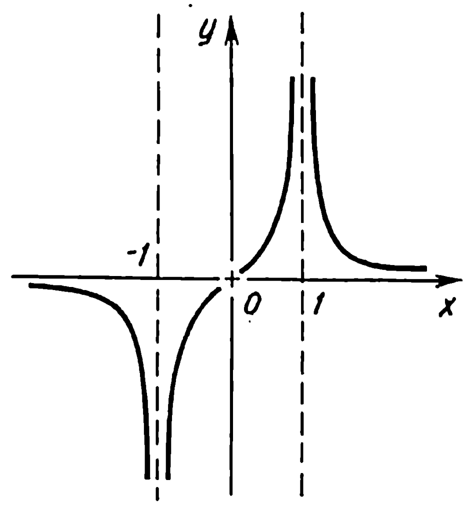
To Ex. 5



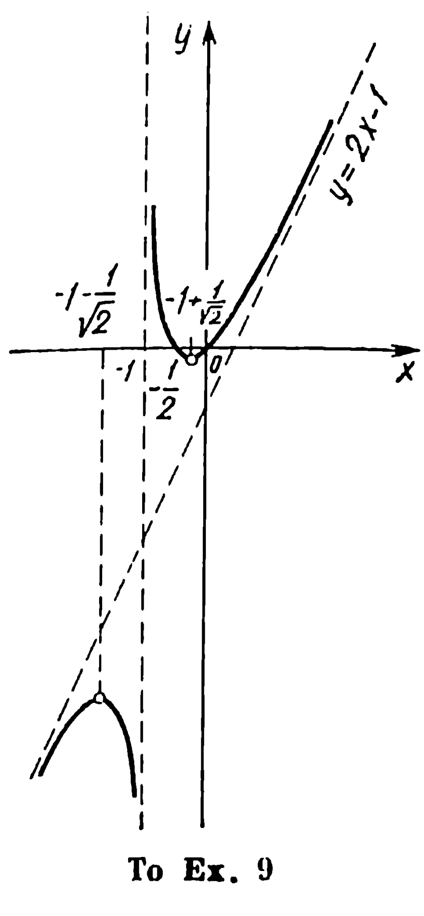
To Ex. 6



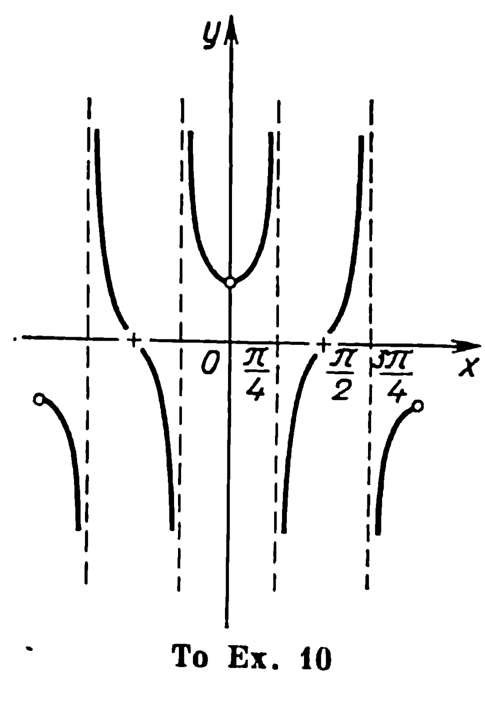
To Ex. 7



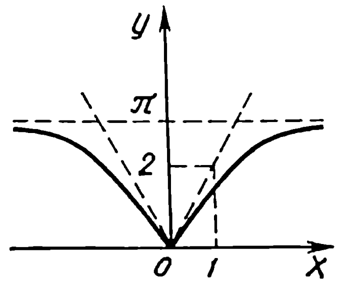
To Ex. 8



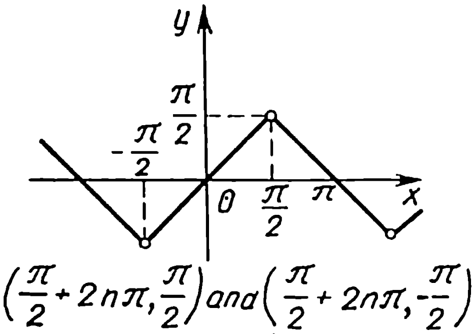
To Ex. 9



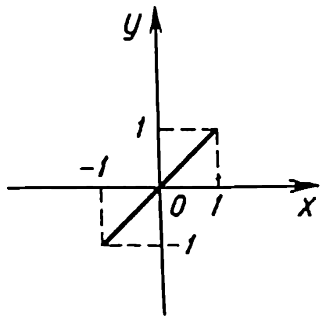
To Ex. 10



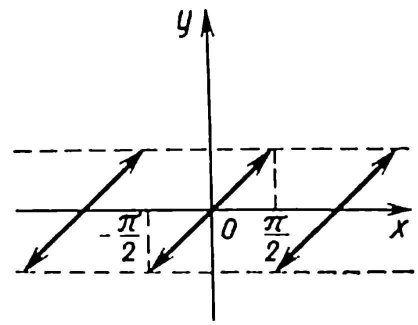
To Ex. 11



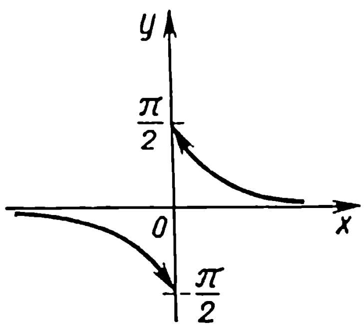
To Ex. 12



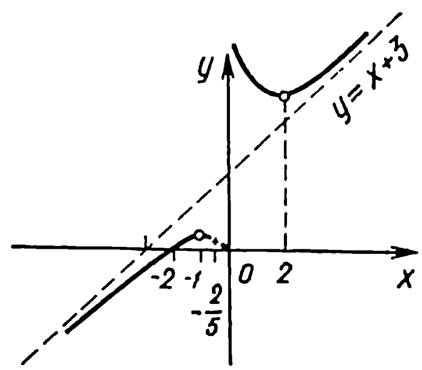
To Ex. 13



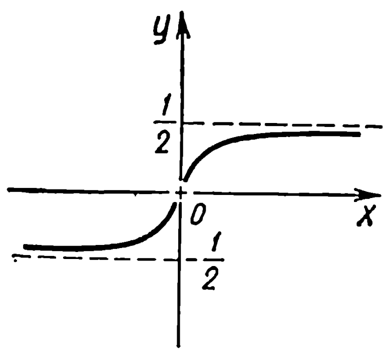
To Ex. 14



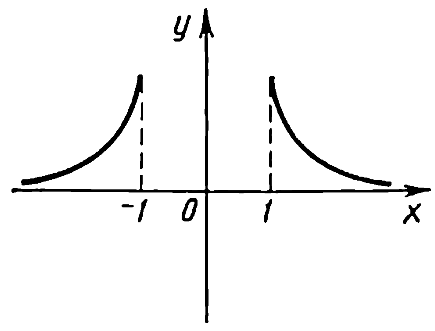
To Ex. 15



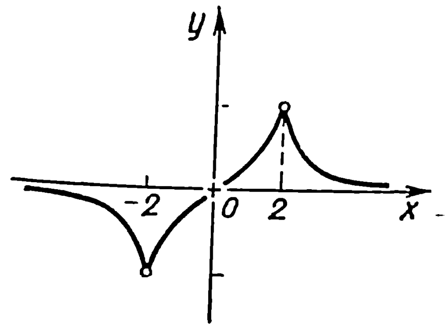
To Ex. 16



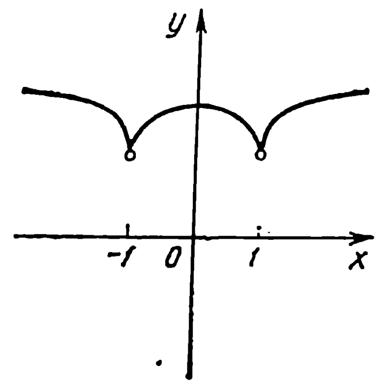
To Ex. 17



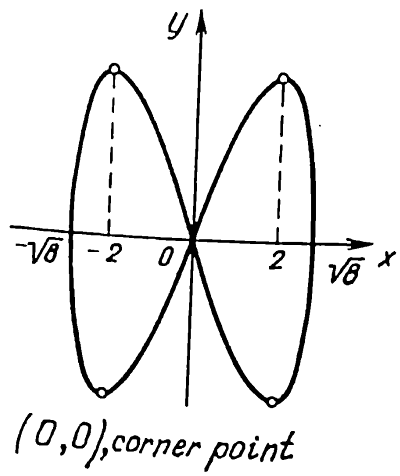
To Ex. 18



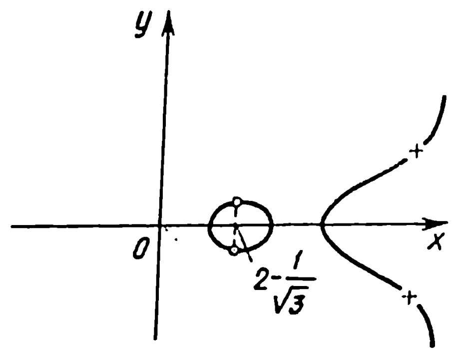
To Ex. 19



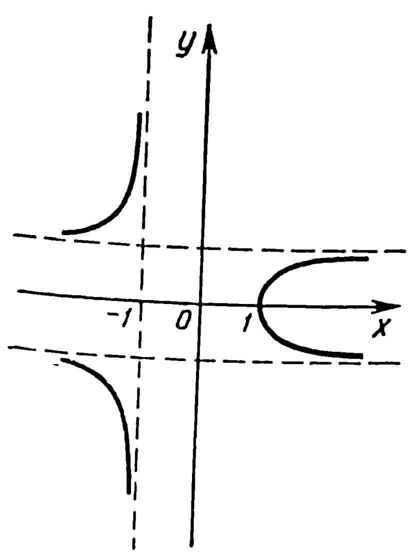
To Ex. 20



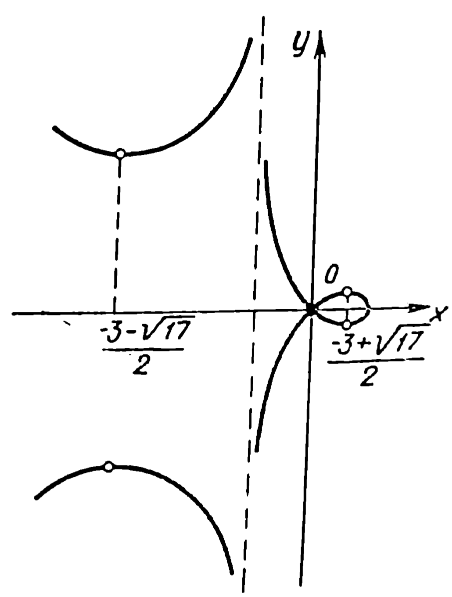
To Ex. 21



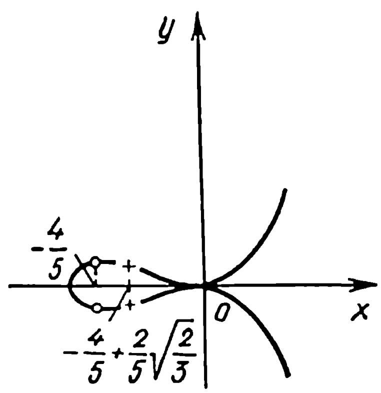
To Ex. 22



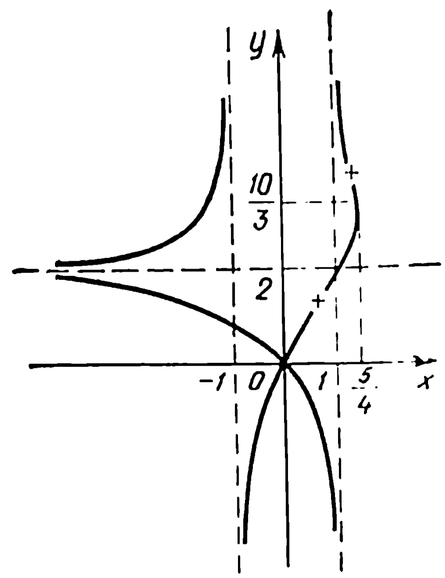
To Ex. 23



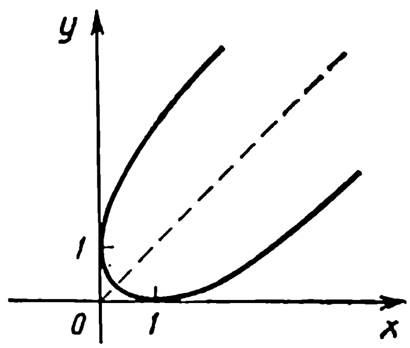
To Ex. 24



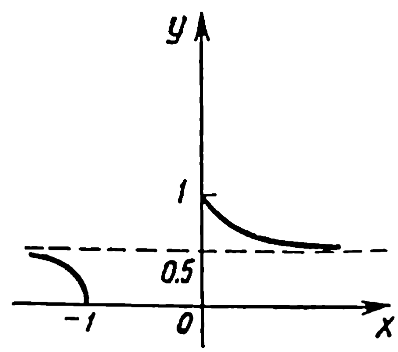
To Ex. 25



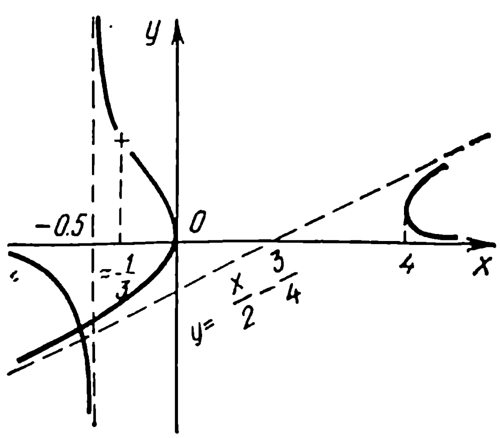
To Ex. 26



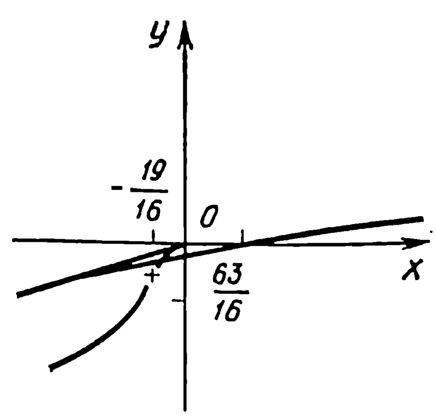
To Ex. 27



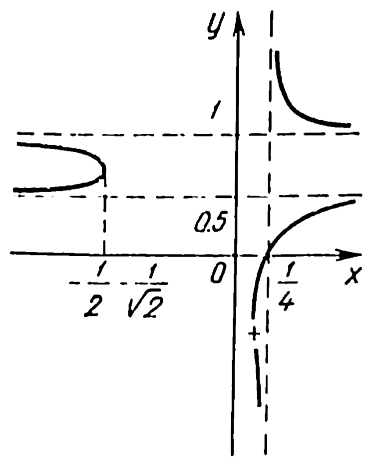
To Ex. 28



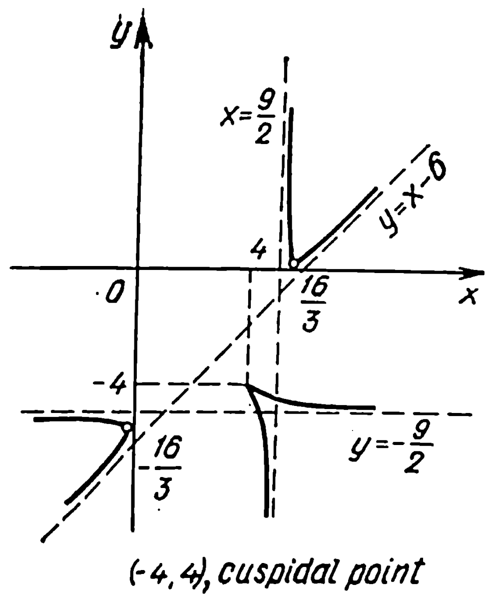
To Ex. 29



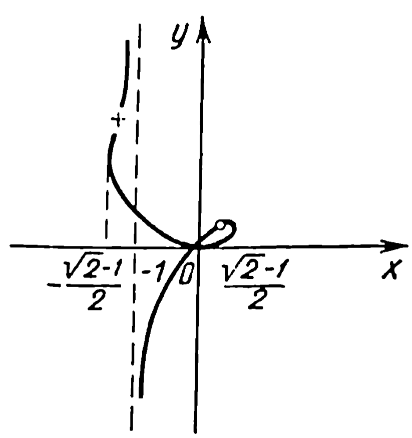
To Ex. 30



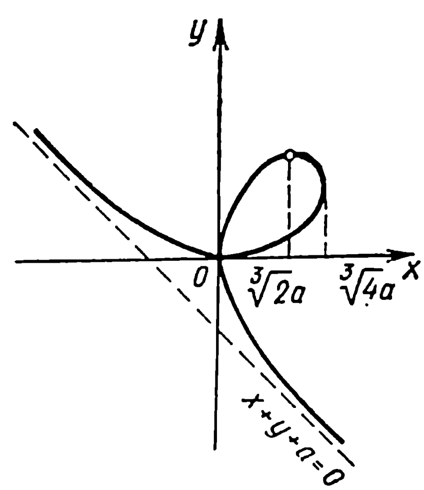
To Ex. 31



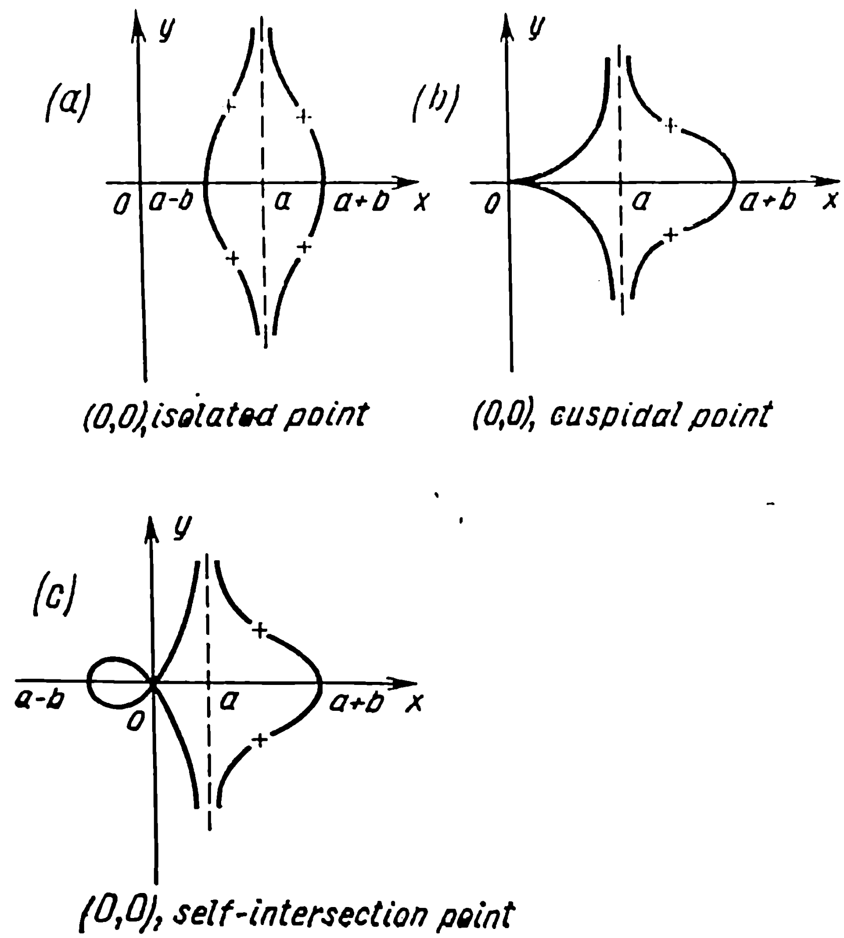
To Ex. 32



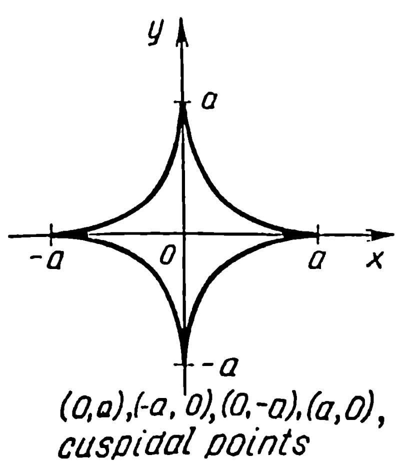
To Ex. 33



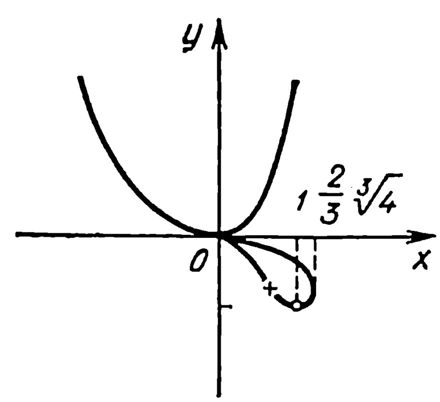
To Ex. 34



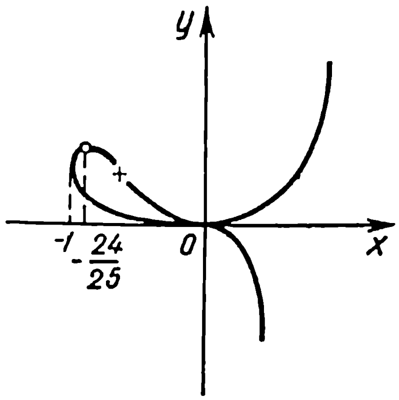
To Ex. 35



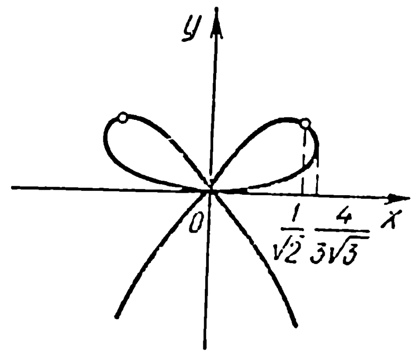
To Ex. 36



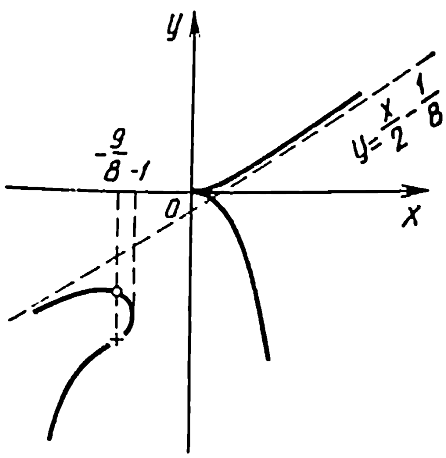
To Ex. 37



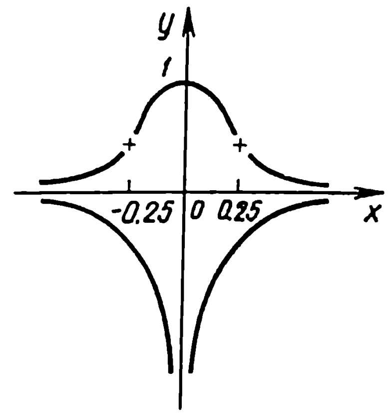
To Ex. 38



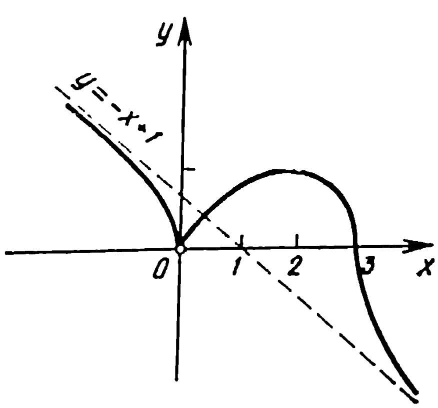
To Ex. 39



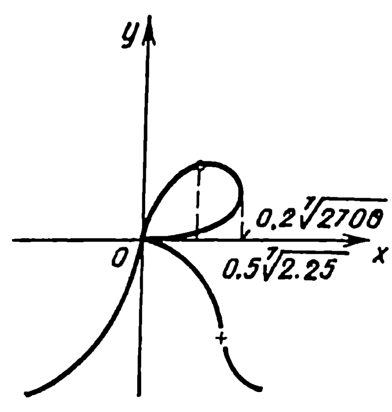
To Ex. 40



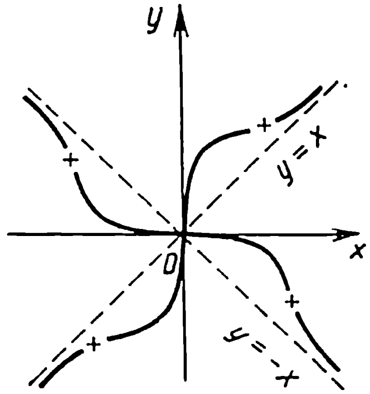
To Ex. 41



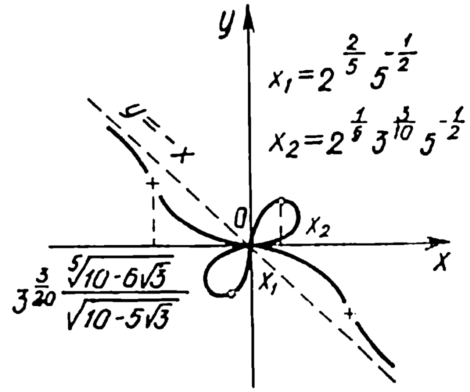
To Ex. 42



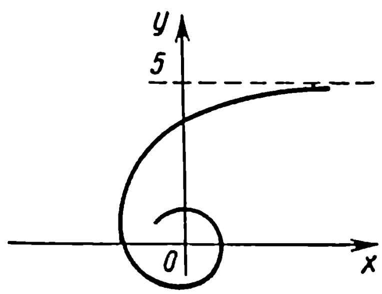
To Ex. 43



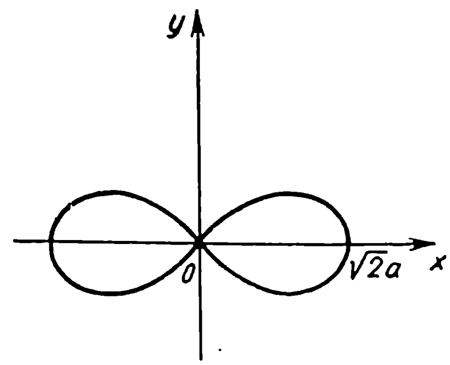
To Ex. 44



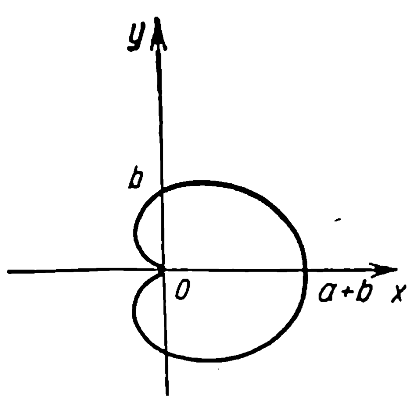
To Ex. 45



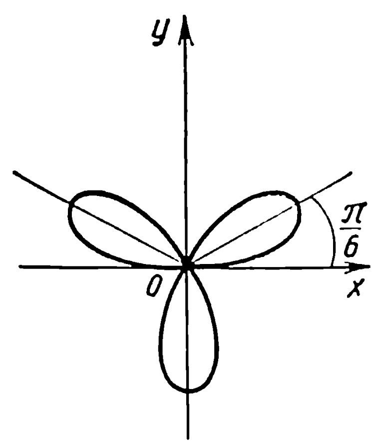
To Ex. 46



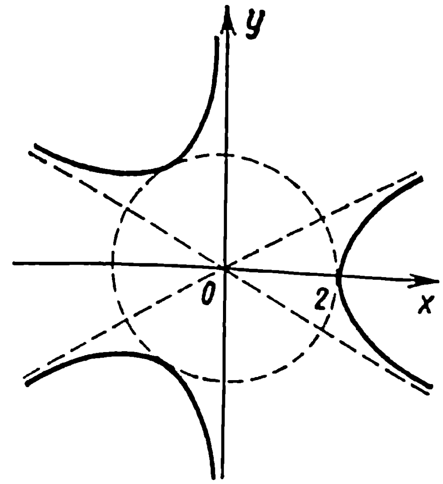
To Ex. 47



To Ex. 48



To Ex. 49



To Ex. 50

$c_2 \in (a, b)$ such that $g'(c_1) > 0$ but $g'(c_2) < 0$ (explain why), whence $f'(c_1) > \frac{f(b)-f(a)}{b-a}$, but $f'(c_2) < \frac{f(b)-f(a)}{b-a}$. This means that at one of the points c_i we have $|f'(c_1)| > \left|\frac{f(b)-f(a)}{b-a}\right|$, i.e. $|f(b)-f(a)| < |f'(c_1)| \cdot |b-a|$. 40. ● Use the result of Exercise 39. 41. 0. 42. 0. 43. α/β. 44. -2. 45. 1. 46. 1/2. 47. $-1/3$. 48. 1. 49. 0. 50. 1. 51. 1/6. 52. 0. 53. $a^a(\ln a - 1)$. 54. $e^{2/\pi}$. 55. 1/e. 56. 1. 57. $e^{-1/6}$. 58. $e^{-1/3}$. 59. 1/2. 60. 1/2. 61. 0. 62. $-e/2$. 63. a^a. 64. 1. 65. (a) $1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^n}{n!}+o(x^n)$, (b) $1+2x+x^2-\frac{2}{3}x^3-\frac{5}{6}x^4-\frac{1}{15}x^5+o(x^5)$, (c) $x-\frac{x^3}{3}+o(x^3)$, (d) $1-\frac{x^2}{2}+\frac{5}{24}x^4+o(x^4)$, (e) $-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}+o(x)^6$, (f) $x-\frac{x^7}{18}-\frac{x^{13}}{3240}+o(x^{13})$, (g) $a+\frac{x}{na^{n-1}}-\frac{(n-1)x^2}{2n^2a^{2n-1}}+o(x^2)$, (h) $1-\frac{1}{2}x+\frac{7}{8}x^2+\frac{7}{16}x^3+o(x^3)$. 66. (a) $1+2(x-1)+(x-1)^2$, (b) $1+(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3+o((x-1)^3)$, (c) $1-\frac{\pi^2}{8}\times(x-1)^2+\frac{\pi^4}{384}(x-1)^4+o((x-1)^4)$. 67. Smaller than 1/3840, (b) smaller than 2×10^{-6}, (c) smaller than 1/16. 68. (a) 2.080, (b) 3.08000, (c) 0.3090, (d) 0.01745241, (e) 0.095, (f) 1.22140, (g) 0.99452. 69. (a) $-1/12$, (b) -2, (c) 1, (d) $-1/4$, (e) 1/2, (f) 0, (g) 1/3, (h) 19/90.

Chapter 7

On the graphs of the functions the bending points are denoted by circles (○) and the points of inflection by crosses (× or +).

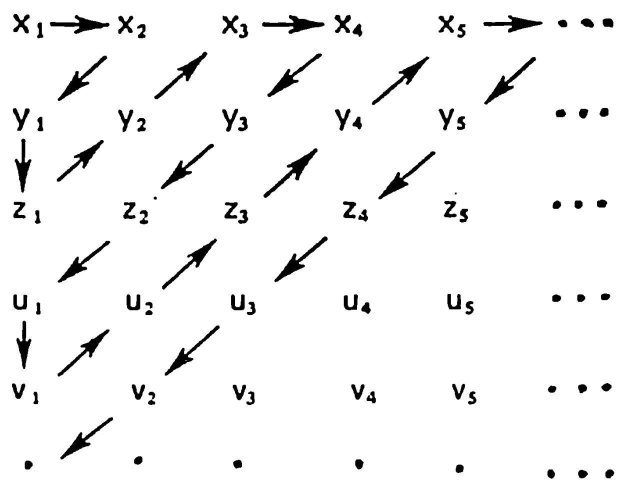
Chapter 8

1. (a) $s=\sum_{i=1}^{n} m_i\Delta x_i=\sum_{i=1}^{n}\left(2+\frac{i-1}{n}\right)^3\frac{1}{n}$, $S=\sum_{i=1}^{n} M_i\Delta x_i=\sum_{i=1}^{n}\left(2+\frac{i}{n}\right)^3\frac{1}{n}$, (b) $s=\sum_{i=1}^{n} 2^{(i-1)/n}\frac{10}{n}$, $S=\sum_{i=1}^{n} 2^{i/n}\frac{10}{n}$. 2. (a) 3, (b) $\frac{1}{2}$, (c) $\frac{3^{m+1}-2^{m+1}}{m+1}$. 4. ● Use the fact that the point $x=0$ is the limit of the points of discontinuity. 6. (a) Yes, it is, (b) no, it is not, (c) no, it is not, (d) yes, it is, with due regard for the remark in item 4 of 8.1, (e) yes, it is, (f) no, it is not. 7. No, yes, no, yes. 9. (a) Yes, (b) no, 10. (a) $2/\pi$, 0, 0, $(2/\pi)\cos\varphi_0$, (b) -1, $-1/3$, $1/2$, 0, 1. 11. 2/3, $(1/15)\,10^{3/2}$, $(1/150)\,100^{3/2}$, (b) 10, (c) $(1/2)\cos\varphi$. 12. $v_{\text{aver}}=(v_0+v_1)/2$, where $v_1=2gh$ is the final velocity of the body. 13. $i_0^2/2$, $i_0^2/2-(i_0^2/2)\times(1/t_0)(T/4\pi)[\sin(4\pi t_0/T+2\varphi)-\sin 2\varphi]$, $i_0^2/2$. 14. (a) 0, (b) $\sin(b^2)$, (c) $-\sin(a^2)$, (d) $2x\sqrt{1+x^4}$, (e) $2x\sqrt{1+x^4}$, (f) $3x^2/\sqrt{1+x^6}-2x/\sqrt{1+x^4}$, (g) $3x^2/\sqrt{1+x^6}-2x/\sqrt{1+x^4}$, (h) $3x^2/\sqrt{1+x^6}$, (i) $3t^2/\sqrt{1+t^{12}}$, (j) 0, (k) $-\int_{t^2}^{x^3} x(x^2+t^4)^{-3/2}\,dx+3x^2/\sqrt{x^2+x^{12}}$.

15. (a) 1, (b) 1, (c) $\pi(2\sin\alpha)$. 16. (a) $\pi\sqrt{2}$, (b) 2/3. 17. 5/6. 18. (a) $0.5\ln(e/2)$, (b) 4π, (c) 1. 19. (a) 1/6, (b) $(1/\sqrt{2})\ln[(9+4\sqrt{2})/7]$, (c) $2-\pi/2$, (d) $\pi^2/4$. 20. No, we cannot. 21. (a) Yes, we can, $\pi/4$, (b) yes, we can, $\pi/4$, (c) no, we cannot, in this interval the hypothesis of Theorem 4 is violated: $0\leqslant\sin t\leqslant 1$. 24. (a) $0.5\ln 3-\pi/2\sqrt{3}$, (b) $(5/27)e^3-2/27$, (c) $4\pi/3-\sqrt{3}$, (d) $2\pi(1/\sqrt{3}-1/2\sqrt{2})$, (e) 1/6, (f) $\pi^3/6-\pi/4$. 27. (a) $\frac{\pi(2m)!\,(2n)!}{2^{2m+2n+1}m!\,n!\,(m+n)!}$, (b) 0 if n is an even number, π if n is an odd number, (c) $\pi/2^n$, (d) $(\pi/2^n)\sin(\pi n/2)$. 28. (a) $(8/27)\times(10\sqrt{10}-1)$, (b) $(e^2+1)/4$, (c) $\ln\tan(\pi/4+a/2)$, (d) $4a[1+\sqrt{3}\ln(1+\sqrt{3})/\sqrt{2}]$, (e) $6a$, (f) $1+[\ln(1+\sqrt{2})]/\sqrt{2}$, (g) $32a$, (h) $\pi a\sqrt{1+4\pi^2}+(a/2)\ln(2\pi+\sqrt{1+4\pi^2})$, (i) $8a$, (j) $3\pi a/2$, (k) 19/3. 30. (a) $(ab/2)[\arcsin(x_1/a)-\arcsin(x_0/a)]-[b/(2a)]\times(x_1\sqrt{a^2-x_1^2}-x_0\sqrt{a^2-x_0^2})$, (b) 9/2, (c) $1/3+2/\pi$, (d) $4a^3/3$, (e) $0.5\coth(\pi/2)$. 31. (a) $(a^3/3)(4\pi^3+3\pi)$, (b) $6\pi a^2$. 32. (a) $3\pi a^2/2$, (b) $\pi a^2/4$, (c) 11π, (d) 2/3. 33. (a) $3a^2/2$, (b) a^2. 34. $(\pi h/6)[(2A+a)B+(A+2a)b]$. 36. (a) $2abc/3$, (b) $4\pi abc/3$, (c) $8\pi abc/3$, (d) $16a^3/3$, (e) $2\pi a^3/3-8a^3/9$. 37. (a) $3\pi ab^2/7$, (b) $16\pi/15$, (c) $8\pi/3$, (d) $\pi^2/2$, (e) $2\pi^2$, (f) $5\pi^2a^3$, (g) $6\pi^3a^3$. 38. $2a^2$, $\pi a^3/2$. 39. $(p^2/8)[\sqrt{2}+5\ln(1+\sqrt{2})]$. 40. $bh^2/6$, $bh^3/12$. 41. $M_x=\pi ab^3/4$, $M_y=\pi a^3b/4$. 42. $3RM/16$. 45. $4\pi^2$. 46. $x_0=(a\sin\alpha)/\alpha$, $y_0=0$. 47. $x_0=9a/20$, $y_0=9a/20$. 48. $x_0=0$, $y_0=0$, $z_0=3a/8$. 49. $\varphi_0=0$, $r_0=5a/6$.

Chapter 9

1. (a) It is sufficient to put every number n from the first set into correspondence with the number $2n$ from the second set, (b) the function $y = (b - a)\,x + a$ realizes a one-to-one correspondence between the elements of the closed intervals $[0, 1]$ and $[a, b]$, (c) the function $y = \tan \dfrac{\pi(2x - a - b)}{2(b - a)}$ realizes a one-to-one correspondence between the elements of the open interval (a, b) and the number line $\mathbf{R}$. 2. (b) The following scheme can be used to enumerate the elements of the union of the countable number of countable sets $\{x_i\}$, $\{y_i\}$, $\{z_i\}$, $\{u_i\}$, $\{v_i\}$, $\ldots$, $\{w_i\}$, $\ldots$:



4. ● (a) Use the method employed in Example 2 in 9.1, (b) use the equality $A = (A \setminus B) + AB$ (see Exercise 5) and the result of Exercise 4 (a), (c) use the result of Exercise 4 (b). 5. (a) Prove that (1°) any element x of the set $(A + B)\,C$ also belongs to the set $(AC + BC)$, (2°) conversely: any element x of the set $(AC + BC)$ also belongs to the set $(A + B)\,C$. 1°. If $x \in (A + B)\,C$, then $x \in (A + B)$ and $x \in C$. Since $x \in (A + B)$, then x belongs to at least one of the sets A and B. Let, for instance, $x \in A$. Then $x \in AC$ and, consequently, $x \in (AC + BC)$. 2°. If $x \in (AC + BC)$, then x belongs to at least one of the sets AC or BC. Let, for instance, $x \in AC$. Then $x \in A$ and $x \in C$. This means that $x \in (A + B)$ and $x \in (A + B)\,C$. 6. We shall prove that any element x of the set $\overline{\bigcup_k A_k}$ also belongs to the set $\bigcap_k \overline{A}_k$. If $x \in \overline{\bigcup_k A_k}$, then $x \notin \bigcup_k A_k$ and, consequently, $x \notin A_k\ \forall k$. Therefore $x \in \overline{A}_k\ \forall k$, and, hence, $x \in \bigcap_k \overline{A}_k$. It remains to prove that any element x of the set $\bigcap_k \overline{A}_k$ also belongs to the set $\overline{\bigcup_k A_k}$. Do it yourself. 7. If we assume that some point $x \in G$ is not an interior point of G, then x will be a limit point of E (explain why) and, consequently, $x \in E$. But this is impossible since $GE = \varnothing$. Thus any point $x \in G$ is an interior point of G, i.e. G is an open set. 8. ● Use the fact that $Q + \overline{Q} = [a, b]$ and $\mu Q = 0$ (see Example 4 in 9.1); hence $\mu \overline{Q} = b - a$.

9 and 10. ● Use the definition of a countable set. 12. ● (a) Use the method employed in Example 1 in 9.2, (b) use the result of (c) in Example 5 in 9.1, (c) use the result of (e) in Example 5 in 9.1. 13. ● Consider the function $y(x) = x$ on the closed interval $[0, 1]$ and use properties 5^0 and 1^0 of measurable functions. 15. The Lebesgue integrability follows from the measurability of $f(x)$ (see Exercise 12a). To prove that $f(x)$ is Riemann integrable on $[0, 1]$, it is sufficient to prove that the lower and upper Darboux integrals are not equal, i.e. $\underline{I} \neq \overline{I}$. Consider the arbitrary partitioning of $[0, 1]$ into subintervals and show that $s = 0$ and $S \geqslant 1 - a$ for it (s and S are Darboux sums). It follows that $\underline{I} = 0, \overline{I} \geqslant 1 - a > 0$. 16. $\int\limits_{[0,1]} f(x)\, d\mu(x) = 1 - a$. 17. $\int\limits_{D} \varphi(x) \times d\mu(x) = (1 - a)/2$. 19. ● Compose a difference $f(x) - g(x)$ and use the result of Exercise 18 for it. 20. ● Use the fact that the set of all lower (upper) Darboux integral sums resulting from partitioning $[a, b]$ into a final number of subintervals is included into the set of all lower (upper) integral sums resulting from partitioning $[a, b]$ into a final number of pairwise nonintersecting measurable sets. 21. Use the fact that for the Lebesgue partition T of the set E there holds an inequality $S_T - s_T \leqslant \delta \cdot \mu E$, where $\delta = \max\limits_{1 \leqslant k \leqslant n} (y_k - y_{k-1})$ and also the fact that $\overline{I} - \underline{I} \leqslant S_T - s_T$. 22. ● To prove the necessity, use the fact that for the Lebesgue partition T of the set E there holds an inequality $S_T - s_T \leqslant \delta \cdot \mu E$, where $\delta = \max\limits_{1 \leqslant k \leqslant n} (y_k, y_{k-1})$. To prove the sufficiency, use the inequality $\overline{I} - \underline{I} \leqslant S_T - s_T$. 23. ● Use the fact that any Lebesgue integral sum $I(E_k, \xi_k)$ of the Lebesgue partition $T = \{E_k\}$ of the set E satisfies the inequalities $s_T \leqslant I(E_k, \xi_k) \leqslant S_T$ and also the fact that $S_T - s_T \leqslant \delta \cdot \mu E$.

Index

Additivity of an integral 190, 239
Antiderivative 112, 196
Argument of a function 53
Asymptotic formulas 73, 76

Basic elementary functions 64
Bernoulli's inequality 22
Binomial theorem 41
Bounded function 136
from above 136
from below 136
sequence 25
set 14
from above 14
from below 14

Cauchy condition for convergence of a sequence 51
Closed interval 12
set 227
Complement of a set 228
Composite function 64
Continuity of a function 64
on the left 63
on the right 63
at a point 63
Convergent sequence 25
series 227
Convexity of the graph downwards 165
upwards 165
Coordinate line 12
Corner point 168
Countable additivity of the Lebesgue measure 228
set 226
Cubable solid 219
Cuspidal point (cusp) 178

Darboux lemma 183
lower integral 183
sum 182
upper integral 183
Decrease of a function on an interval 147
at a point 147
Decreasing sequence 45
Definite integral 181, 189
Derivative 85
left-hand 87
right-hand 87
of order n 106
of a vector function 89
Defference of sets 227
Differentiable function 100
Differential of a function 101
of order n 106
of an independent variable 101
Differentiation 85
Differentiation rules 88
Divergent sequence 25
Divided difference 85
Domain of definition of a function 53

Element of a sequence 24
Elementary function 64
Elliptic integral 212
ε-neighbourhood 12
Equality of real numbers 11
Equivalent functions 235
infinitesimals 69
sets 226

Euler's substitutions 129
Even function 163
Existential quantifier 14
Explicit function 163
Extremum of a function 164

First mean value theorem 148
Formula of Cauchy 149
for a change of variable 117
for integration by parts 121
of Lagrange 148
of Leibniz 105
of Maclaurin 159
of Newton-Leibniz 197
of Taylor 158
Formulas for calculating the length of a plane curve 210
the mass of a plane curve 221
the moments and coordinates 222
the centre of gravity of a plane curve 222
of the volume of a solid 218
of mean value 190
Function 53
argument of 53
basic elementary 64
bounded 136
composite 64
continuity of 63
decrease of 147
derivative of 85
differentiable 100
differential of 101
domain of definition of 53
elementary 64
even 163
explicit 163
greatest lower bound of 137
increase of 147-148
increment of 85
infinitesimal 69
least upper bound of 137
Lebesgue integrable 238
limit of 53
mean value of 191
measurable on a set 235
odd 164
oscillation of 137
parametric representation of 89
particular value of 53
periodic 164
piecewise continuous 183
range of 53
Riemann integrable 182
set of values 53
uniform continuity of 134

General form of a remainder 158
Geometric meaning of a derivative 86
of a differential 101
of Rolle's theorems 148
of uniform continuity of a function 143
Greatest lower bound of a function 137
of a set 15

Half-open (half-closed) interval 12
Hodograph 90
Hyperbolic cosine 64
cotangent 64
sine 64
tangent 64

Increase of a function on an interval 147
at a point 147
Increasing sequence 45
Increment of a function 85
Indefinite integral 112
Independent variable 53
Indeterminate forms 35, 78
Infinite derivative 87
Infinitely differentiable function 105
large sequence 31
small sequence 31
Infinitesimal function at a point 69
sequence 31
Infinitesimals of the same order 69
Integral sum 181
Integration of irrational functions 129

Integration
  of rational functions 125
  of trigonometric functions 134
Interior point of a set 227
Intersection of sets 226
Invariance of the form of the first differential 102

Least upper bound of a function 137
  of a set 15
Lebesgue integral 238
  with variable upper limit 143
Lebesque integrable function 238
  integral sum 239
  lower integral 238
  lower sum 238
  measurable set 228
  partition 239
  upper integral 238
  upper sum 238
Left-hand derivative 87
  limit 54
Length of a curve 210
Limit of a function at a point 53
  as $x \to \infty$ 55
  of integral sums 181
  of length of a polygonal line 210
  of sequences 25
  of vector functions 89
Limit inferior (superior) of a sequence 47
Limit point of a sequence 47
  of a set 53
Linearity of an integral 189, 239
Local extremum 164
  maximum 164
  minimum 164
Lower bound of a function 136
  of a number set 14
  Lebesgue sum 239

Maximum value of a function on a set 137
Mean value formula 190
Method of cancelling 127
  of a change of variable 117-118, 197
  of induction 22
  of integration by parts 121, 197
  of undeterminate coefficients 126
Minimum value of a function on a set 137
Monotonic sequence 45

Neighbourhood 12
Nondecreasing sequence 45
Nonincreasing sequence 45
Normal 91
Number e 64
Number line 12
  sequence 24
  series 227

"o small" 69
Oblique asymptote 163
Odd function 164
One-to-one correspondence 226
Open interval 12
  set 227
Oscillation of a function 137
Outer measure 228

Parameter 89
Parametric representation of a function 89
Partial fractions 125
  sum of a series 227
Particular value of a function 53
Partition of a set 238
Period of a function 164
Periodic function 164
Physical meaining of a derivative 86
  of a differential 101
  of Lagrange's theorem 148
  of Rolle's theorem 148
  of a vector function and its derivative 89-90
Piecewise continuous function 183
Plane figure 213
  area of (in the sense of Jordan) 213

Point of discontinuity 64
of the first kind 65
of the second kind 65
of inflection 165
of the number line 12
of a possible extremum 164
of removable discontinuity 64
Product of real numbers 19

Quadrable plane figure 213

Range of a function 53
Rectifiable curve 210
Remainder of an asymptotic formula 73
of Taylor formula in Cauchy's form 159
in Lagrange's form 159
in Peano's form 158
Remarkable limits 61, 64
Riemann integrable function 182
Right-hand derivative 87
limit 54
Rule of comparison of real numbers 11
of constructing a negation 15
of L'Hospital 154-155
Rules of differentiation 88

Scheme of constructing a curve represented parametrically 173-175
the graph of a function $y = f(x)$ 165
Sequence 24
bounded 25
convergent 25
decreasing 45
divergent 25
element of 24
increasing 45
infinitely large 31
infinitesimal 31
limit of 25
limit point of 47
monotonic 45
strictly monotonic 45
term of 24
unbounded 25
Set 14
bounded 14
closed 227
complement of 228
countable 226
greastest lower bound of 15
least upper bound of 15
Lebesgue measurable 168
limit point of 39
lower bound of 14
open 227
of the power of continuum 226
upper bound of 14
Sets of the same power 226
Simple closed curve 209
open curve 209
Solid 218
Strictly monotonic sequence 45
Subsequence 47
Sum of real numbers 18
of series 227
of sets 226

Table of derivatives of basic elementary functions 86
of fundamental indefinite integrals 114-115
Tangent 87
Taylor's polynomial 158
Term of a sequence 24
of a series 227
Theorem of Bolzano-Weierstrass 47
of Cantor 143
of Cauchy 149
on the constancy of sign of a continuous function 136
on continuity of a composite function 64
on the convergence of a monotonic bounded sequence 45
on the decomposition of a rational fraction into the sum of partial fractions 125
on the derivative of a composite function 88
of an inverse function 88
on the existence of the least upper bound and the greatest lower bound of a set 15

Theorem
of Lagrange 148
on local boundedness of a function continuous at a point 136
of Lusin 236
of Rolle 148
on the structure of open sets 227
on three sequences 36
of Weierstrass, the first 136
the second 137
Theorems on antiderivatives 112, 117, 121, 196-197
on continuous functions 64, 136
on derivatives 88, 89
on differentiable functions 101, 147-148
on the direction of convexity at the point of inflection of the graph of a function 165
on the extremum of a function 164
on Lebesgue integrable functions 238-239
on limits of functions 55
of sequences 25, 36, 47, 51
on measurable functions 235
on Riemann integrable function 183-184, 190

Unbounded sequence 25
Uniform continuity of a function on a set 142
Union of sets 226
Universal quantifier 14
Upper bound of a function 136
of a number set 14
integral of Darboux 183
of Lebesgue 238
Lebesgue sum 238
limit of a sequence 47

Variable arc 210
Vector of a normal 91
Vector function 89
Vertical asymptote 163
Volume of a solid (in the sense of Jordan) 218